MAT 351: Partial Differential Equations
Assignment 3, due Oct. 3, 2016

Summary:

A **conservation law** is a first order PDE of the form

$$u_t + \partial_x A(u) = 0 \, .$$

We think of x as a spatial variable, and t as time. The function u is interpreted as the density of a (one-dimensional) fluid, and A describes the flow speed as a function of density. In gas dynamics, A is usually increasing and convex, while in traffic modeling it is increasing and concave. The simplest example is **Burger's equation**, where $A(u) = \frac{1}{2}u^2$, resulting in

$$u_t + uu_x = 0 \, .$$

The results we discuss are representative for scalar conservation laws in one dimension. Systems of conservation laws in higher spatial dimensions, which appear in fluid dynamics, pose greater challenges that are beyond the scope of this course.

Conservation laws are examples of **quasilinear** equations, that is, equations that are linear in the highest order derivatives, with coefficients that depend on the unknown function. Solutions canbe constructed by solving the characteristic equations

$$\begin{aligned} \dot{x} &= a(z) \\ \dot{z} &= 0 \, , \end{aligned}$$

where x, z are real variables, and $a(z) = A'(z)$ is the derivative of A. What makes conservation laws interesting is that singularities can develop after a finite time, even when the initial values are smooth. This motivates the following definition.

- A function $u = u(x, t)$ is called a **weak solution** of the conservation law, if

  $$\int_{\partial D} \begin{pmatrix} A(u) \\ u \end{pmatrix} \cdot n = 0$$

  for every smooth domain $D \subset \mathbb{R}^2$. Here, $n = n(x, t)$ is the outward unit normal to D at a boundary point (x, t).

The prototypical singularity of a weak solution is a **shock**, where the value of the solution jumps across a smooth curve $x = \gamma(t)$. The speed of the shock is determined by the

- **Rankine-Hugoniot condition:** $\gamma'(t) = \dfrac{A(u_\ell) - A(u_r)}{u_\ell - u_r}$ at every point $(\gamma(t), t)$ on the curve.

For Burger's equation, the shock speed is just the average of the characteristic speeds immediately to the left and right of the shock. We can think of the characteristic ODE as an infinitesimal version of the Rankine-Hugoniot condition.

It turns out that weak solutions at a given initial-value problem are not unique. Uniqueness is restored by requiring additionally that the solution satisfy

- **Lax' entropy condition:** At a shock, $a(u_\ell) > \gamma'(t) > a(u_r)$.

This means that nearby characteristics should always run *into* the shock. Characteristics emanating from a shock are viewed as unphysical. Note that the entropy condition breaks the symmetry of the PDE under the change of variables $(x,t) \to (-x,-t)$, and introduces a preferred direction of time. This is reminiscent of the second law of thermodynamics, which says that entropy always increases with time. One consequence is that shocks travel forward in gas dynamics, but backward in traffic modeling. A weak solution that satifies Lax' condition is called an **entropy solution**.

We note in passing that the method of characteristics can be adapted to solve first order fully nonlinear equations locally, i.e., in a neighborhood of the initial data. Examples of fully nonlinear first order equations are the eikonal equation $|\nabla u| = 1$ (which describes characteristic surfaces for the wave equation), and the Hamilton-Jacobi equation $u_t + H(u, \nabla u) = 0$ (which appears in classical mechanics).

Assignments:

Read Chapter 14.1.

Hand-in (due Monday, October 3):

9. *(Problem 1.5 of Pinchover-Rubinstein)*
   Let $p : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Prove that the equation
   $$u_t = p(u)u_x$$
   has a solution that satisfies the functional relation $u(x,t) = f(x + p(u)t)$, where f is a differentiable function. In particular, find such solutions for the following equations:
   (a) $u_t = ku_x$, where k is a constant; (b) $u_t = uu_x$; (c) $u_t = u \sin u u_x$.

10. Let u be an entropy solution of the conservation law
    $$u_t + \partial_x A(u) = 0\,,$$
    where A is a non-decreasing function on the real line with $A(0) = 0$. Justify the following statement: If A is convex, then shocks travel forward (to the right); if A is concave, they travel backward (to the left).

11. A **multi-index** is a vector $\alpha = (\alpha_1, \dots, \alpha_n)$ of nonnegative integers. Define
    - $|\alpha| = \alpha_1 + \cdots + \alpha_n$, the **order** of α,
    - the power $x^\alpha = x_1^{\alpha_1} \cdot \cdots \cdot x_n^{\alpha_n}$,
    - the **factorial** $\alpha! = \alpha_1! \cdot \cdots \cdot \alpha_n!$ and the **multinomial coefficient** $\binom{|\alpha|}{\alpha} = \frac{|\alpha|!}{\alpha!} = \frac{|\alpha|!}{\alpha_1! \cdot \cdots \cdot \alpha_n!}$.

    (a) Show that $(x_1 + \cdots + x_n)^k = \displaystyle\sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} x^\alpha$. (*Hint:* Induction over either k or n).

    (b) State Taylor's formula of order k for a smooth function of n variables (without proof).