MAT 351: Partial Differential Equations
Sept. 11, 2017

Summary

A **partial differential equation** (PDE) is an equation of the form

$$F(x, u, Du, ..., D^k u) = 0\,, \tag{1}$$

where

- $u = u(x_1, \dots, x_n)$ is the unknown function, also called the **dependent variable**; (u and F could be vector-valued; in that case, Eq. (1) is called a **system** of PDE);
- the integer $k \geq 1$ is called the **order** of the PDE;
- $D^k u$ means all the k-th order partial derivatives of u; and
- the **independent variables** $x = (x_1, \dots, x_n)$ range over some open set $D \subset \mathbb{R}^n$. (It is usually assumed that D is connected.)

A **solution** of the PDE is a function $u(x_1, \dots, x_n)$ that satisfies Eq. (1) for all $x \in D$.

We will mostly consider first- and second order equations in two, three, or four variables. Ideally, one would like to represent the solution of a given PDE explicitly in terms of its boundary values. It turns out that this is possible only for a small number of classical equations Among these are the **transport equation** $u_t + bu_x$=0, **Poisson's equation** $u_{xx} + u_{yy} = f(x, y)$, the **heat equation** or **diffusion equation** $u_t = u_{xx}$, the **wave equation** $u_{tt} - u_{xx} = 0$, and the **Schrödinger equation** $iu_t + u_{xx} - V(x)u = 0$, all of which are fundamental in Physics. These equations are all **linear**, i.e., they can be written as

$$Lu = f\,,$$

where L is a linear differential operator and f is a given function.

Almost nothing can be said about a general non-linear PDE. Fundamental questions are:

- Does there **exist** a solution for a given PDE?
- Under what additional boundary conditions is the solution **unique**?
- Does the solution depend **continuously** on the data?

If the answer is "Yes!" to all three questions, then the boundary-value problem is **well-posed**. Well-posedness is crucial, if one wants to evaluate the solution of a PDE numerically, because in an ill-posed problem, even small discretization errors can have a devastating effect.

Once existence and uniqueness of solutions have been establihsed for a paeticular PDE, it is often possible to ensure continuous dependence on data by choosing a suitable function space. But there is no reason for a general PDE to be well-posed. There even is a famous example of a linear PDE that has no solutions!

Assignments

Please read Chapter 1 of Strauss.
First tutorial this week: Friday, September 15, 10:10am in RW 142.

Problems for discussion:

1. *(Classification of PDE)*
   For each of the following PDE, what is its order?
   Is it linear? If yes, is it homogeneous or not? (Why?)

   (a) $u_t - u_{xx} + 1 = 0$;

   (b) $u_x + e^y u_y = 0$;

   (c) $u_t + uu_x + u_{xxx} = 0$;
   (Korteveg-de-Vries equation)

   (d) $u_{xx} + u_{yy} + u_{zz} = \lambda u$, where λ is a constant;
   (eigenvalue problem for the Laplacian)

   (e) $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$;
   (Laplace's equation in polar coordinates)

   (f) $u_x u_y = 0$;

   (g) $u_t + (u_x)^2 = 0$;
   (a Hamilton-Jacobi equation)

   (h) $u_t + (u^2)_x = 0$;
   (a conservation law)

2. *(Constructing an explicit solution)*
   By trial and error, find a solution of the heat equation $u_t = u_{xx}$ with initial condition $u(x, 0) = x^2$.

3. (a) Find the value of a such that

$$\begin{aligned} u_x &= ax^2y + y \\ u_y &= x^3 + x \end{aligned}$$

   has a solution. For this value of a, determine the unique solution that satisfies the initial condition $u(0, 0) = 0$.

   (b) What is the general solution of the above system of equations?
   Is the initial-value problem well-posed?

4. *(Method of characteristics)*
   Solve the equation $yu_x + xu_y = 0$ with $u(0, y) = \cos y$. Please sketch some characteristics! In what region of the plane is the solution uniquely determined? If you enlarge the region, what fails – existence or uniqueness?