MAT 351: Partial Differential Equations
Assignment 9 — December 1, 2017

Poisson's equation

$$\Delta u = f$$

describes physical systems in equilibrium such as steady states for diffusion equations (with f describing sources and sinks), Maxwell's equation for a static electric field (with charge distribution given by f), and expected values of many interesting random variables under Brownian motion starting at x. It is the prototype of an **elliptic** equation.

The most important case is **Laplace's equation**

$$\Delta u = 0$$

whose solutions are called **harmonic** functions. In one dimension, the only harmonic functions are the linear, $u(x) = ax + b$. In two dimension, every harmonic function is the real part of a holomorphic function. As such, it is **analytic** (i.e., smooth, and agrees with its Taylor series), and satisfies the **strong maximum principle** (i.e., u cannot assume a local maximum or minimum on a connected domain unless it is constant). We will show that harmonic functions in higher dimensions share the last two properties. A useful fact is the **mean value property**, which says that the average of a harmonic function over a ball or a sphere agrees with its value at the center.

We will discuss methods for solving Poisson problems, consisting of Poisson's equation on a domain together with specified boundary values. The goal is to recover u from f and the given boundary data. In the special case of Laplace's equation on the unit disc, **Poisson's formula**

$$u(r\cos\theta, r\sin\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\phi-\theta)+r^2}\, g(\phi)\, d\tilde{\theta}$$

provides the unique harmonic function u on the disc with boundary values $u(\cos\theta, \sin\theta) = g(\theta)$.

The **Poisson kernel**

$$P_r(\theta) = \frac{1}{2\pi}\cdot\frac{1-r^2}{1-2r\cos\theta+r^2}$$

has the properties that

- $P_r(\theta) > 0$ for all $0 \le r < 1$ and all θ;
- $\int_0^{2\pi} P_r(\theta)\, d\theta = 1$ for all $0 \le r < 1$;

Thus, Poisson's formula represents the solution $u(x, y)$ as an average of the boundary data, weighted by the Poisson kernel P_r, where $r^2 = x^2 + y^2$. As the point $(x, y) = (r\cos\theta, r\sin\theta)$ approaches the boundary of the disc, the weight in Poisson's formula concentrates at $\phi = \theta$,

- $\lim_{r\to 1} P_r(\theta) = 0$ for all $\theta \neq 0$.
  In fact, for every $\delta > 0$, the convergence is *uniform* on $[-\pi, \pi] \setminus (-\varepsilon, \varepsilon)$.

In other words, P_r approximates the δ-distribution, in the sense that

$$\lim_{r\to 1}\int_0^{2\pi} P_r(\theta)h(\theta)\,d\theta = h(0)$$

for every continuous 2π-periodic function h. By definition, the **delta distribution** is the linear transformation that sends the function h to its value at zero,

$$\delta(h) = \int_0^{2\pi} \delta(\theta)h(\theta)\,d\theta = h(0).$$

Remark. Poisson's formula is analogous to the integral representation of solutions to the heat equation on the real line in terms of the initial data (see Assignment 4, top of second page). The Poisson kernel plays the role of the fundamental solution, and the limit $r \to 0$ in Poisson's formula corresponds to the limit $t \to 0$ in the heat equation.

Read: Chapters 6.1 and 6.3 of Strauss.

Hand-in (due Friday, January 5):

(H1) Solve $u_{xx} + u_{yy} = 0$ in the unit disk with the boundary condition $u = \sin^3\theta$.

(H2) Derive Poisson's formula for the exterior of the unit disc ($r > 1$).

For discussion and practice:

1. Suppose that u is a harmonic function in the disk $D = \{r < 2\}$ in two dimensions, and that $u = 2016 + (\sin\theta)^{17}$ for $r = 2$. Without computing the solution, find
(a) the maximum of u on D;
(b) the value of u at the origin;
(c) the integral of u over the disk.

2. (a) Expand the function $\phi(x) = |\sin x|$ as a cosine series on $[-\pi, \pi]$.
(b) Find the sums $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}$.

3. Find the sums of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$.

4. Let f be a smooth 2π-periodic function with $\int_{-\pi}^{\pi} f(x)\,dx = 0$. Use the Fourier series representation and Parseval's identity to show that $||f|| \le ||f'||$.

5. Find the radial solutions (depending only on $r = |x|$) of the equation $u_{xx} + u_{yy} + u_{zz} = k^2 u$, where k is a positive constant. (*Hint:* Substitute $u(r) = \frac{v(r)}{r}$. Solutions may blow up at $r = 0$.)