MAT 351: Partial Differential Equations
Assignment 10 — January 9, 2018

The properties of harmonic functions can be understood by analogy with holomorphic functions. Holomorphic functions in two real variables are in fact precisely the real parts of holomorphic functions, and their properties follow directly from the corresponding properties of holomorphic functions. In particular, Poisson's formula on the disc follows from Cauchy's integral formula. The analogy persists in higher dimensions, but the properties require different proofs.

Here are a few consequences of Poisson's formula:

1. **Completeness of the standard Fourier basis.** The functions $(2\pi)^{-1/2}(e^{ikx})_{k\in\mathbb{Z}}$ form an orthonormal basis for $L^2(-\pi,\pi)$.

2. **Mean value property.** If u is harmonic on a domain $D\subset\mathbb{R}^n$, then its value at a point x equals its average over any ball $B_r(x)$ contained in D; it also agrees with the average over the boundary sphere $\partial B_r(x)$.

3. **Strong maximum principle.** If a harmonic function u on D assumes its maximum or minimum in the interior of D, then u is constant.

   (In two dimensions, this follows from the Open Mapping Theorem.)

4. **Smoothness.** Harmonic functions are smooth, in fact, real-analytic.

5. **Growth.** If a harmonic function on $\mathbb{R}^n$ is bounded, then it is constant.

   (In two dimensions, this follows from Liouville's theorem.)

6. **Unique continuation.** If two harmonic functions on a connected domain agree on an open subset, then they agree on the entire domain.

7. If a continuous continuous function has the mean value property, then it is harmonic.

   (In two dimensions, this follows from Morera's theorem.)

Read: Chapter 7.1 of Strauss.

Hand-in (due Friday, January 12):

(H1) Let D be the unit disc in the plane, and denote by D_+ its intersection with the half-space $y>0$. Let u be a harmonic function D_+ that is continuous on the closure $\overline{D_+}$. Assume that u vanishes on the flat part of the boundary $\{(x,0)\mid -1\le x\le 1\}$, and extend it to a function $\tilde{u}$ on the whole disc by odd reflection,

$$\tilde{u}(x,y)=\begin{cases} u(x,y)\,, & (x,y)\in\overline{D}\,, y\ge 0\\ -u(x,-y) & (x,y)\in\overline{D}, y\le 0\,.\end{cases}$$

Prove that $\tilde{u}$ is harmonic on D, in two ways:

(a) Show directly that $\Delta\tilde{u}(x,y) = 0$ when $y = 0$.
Note: You need to assume here that the second derivatives of u are continuous on $\overline{D}_+$.

(b) Identify $\tilde{u}$ as the solution of a suitable boundary-value problem.
Hint: You will need to use uniqueness twice.

(H2) Solve $\Delta u = 0$ in the spherical shell $0 < a < r < b$ in $\mathbb{R}^n$ for $n \geq 2$ with the boundary conditions $u = A$ on $r = a$ and $u = B$ on $r = b$. (*Hint:* Look for a radial solution. Your formula will look different for $n = 2$ than in higher dimensions).

For discussion and practice:

1. Use the Mean Value Property of harmonic functions in n variables to derive the maximum principle. Conclude that the solution of Poisson's problem $\Delta u = f$ on a bounded domain D, with Dirichlet boundary conditions $u|_{\partial D} = g$ is unique (assuming it exists).

2. Let D be an open set with smooth boundary in $\mathbb{R}^n$. Use the divergence theorem to show that the Neumann problem
$$\Delta u = f \text{ in } D\,, \quad \nabla u \cdot \nu = g \text{ on } \partial D$$
cannot have a solution unless $int_D f\,dx = \int_{\partial D} g\,dS$.

3. *Dirichlet's principle for Neumann boundary conditions (Strauss, Problem 7.1.5)*
Prove that among *all* real-valued functions w on D, the quantity
$$E(w) = \frac{1}{2}\int_D |\nabla w|^2\,dx - \int_{\partial D} hw\,ds$$
is minimized by $w = u$, where u is a harmonic function that satisfies the Neumann boundary condition $\nabla u \cdot n|_{\partial D} = h$. Here, h is a given function on ∂D with $\int_{\partial D} h\,dS = 0$.

4. Consider a homogeneous polynomial in two variables
$$P(x,y) = a_0x^k + a_1x^{k-1}y + \cdots + a_ky^k\,.$$
(a) Under what conditions on the coefficients is the polynomial harmonic? How many linearly independent harmonic polynomials are there of degree k?

(b) Write down a basis of the space of harmonic polynomials of degree $k \leq 4$, in both Cartesian and polar coordinates. Identify them as the real (or imaginary) parts of holomorphic functions.

5. *Spherical harmonics in three variables*
How many linearly independent homogeneous polynomials of degree k are there in three variables? How many linearly independent *harmonic* homogeneous polynomials of degree k are there? (*Hint:* Write the polynomial as
$$P(x,y,z) = \sum_{j=0}^{k} p_j(x,y)z^{k-j}\,,$$
where each p_j is a homogeneous polynomial of degree j in two variables. Consider the Laplacian as a linear transformation that maps polynomials of degree k to polynomials of degree $k-2$. You may assume that this map is onto.)