MAT 351: Partial Differential Equations
Assignment 11 — January 12, 2018

Summary

The **fundamental solution** of the Laplacian in $\mathbb{R}^n$ is defined by

$$\Phi(x) = \begin{cases} \frac{1}{2\pi}\log|x|\,, & n = 2\,, \\ -\frac{1}{n(n-2)\omega_n|x|^{n-2}}\,, & n \geq 3\,, \end{cases}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. In three dimensions $\Phi(x) = -\frac{1}{4\pi|x|}$ can be interpreted as the gravitational potential of a point mass, or equivalently, the electrostatic field of a point charge at the origin.

If f is a bounded function on $\mathbb{R}^n$ (where $n \geq 3$) that vanishes outside some ball, then

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)\,dy$$

is the unique solution of Poisson's equation

$$\Delta u = f \quad \text{on } \mathbb{R}^n$$

with $u(x) \to 0$ as $|x| \to \infty$. (Normalizing the potential to vanish at infinity is a standard choice in Physics. There are many other solutions of Poisson's equation, all of which which grow at infinity.) We say that "$\Delta\Phi = \delta$ in the sense of distributions".

A similar formula holds for Poisson's equation on a bounded domain $D \subset \mathbb{R}^n$. The **Green's function** $G(y,x)$ for D is defined by the properties that for every fixed $x \in D$,

- $G(y,x) - \Phi(x,y)$ is smooth and harmonic in y for $y \in D$;
- $G(y,x) = 0$ for $y \in \partial D$.

Then the unique solution of the Dirichlet problem

$$\begin{aligned} &\Delta u = f \quad \text{on } D\,, \\ &u(x) = g(x) \quad \text{for } x \in \partial D \end{aligned}$$

is given by

$$u(x) = \int_D G(y,x)f(y)\,dy + \int_{\partial D} g(y)\nabla_y G(y,x)\cdot N(y)\,dS(y)\,.$$

We will see that the Green's function is negative and symmetric,

- $G(x,y) < 0$ for all $x, y \in D$ with $x \neq y$;
- $G(x,y) = G(y,x)$.

The function defined on the boundary by

$$P(x,y) = \nabla_y G(x,y) \cdot N(y) \quad \text{for } y \in \partial D$$

is called the **Poisson kernel** associated with D.

The proofs are based on **Green's identities:** For any pair of smooth functions u, v on D, we have

$$\int_D v\Delta u \, dx = -\int_D \nabla u \cdot \nabla v \, dx + \int_{\partial D} v \nabla u \cdot N(x) \, dS(x)\,, \tag{1}$$

$$\int_D (u\Delta v - v\Delta u) \, dx = \int_{\partial D} (u\nabla v - v\nabla u) \cdot N(x) \, dS(x)\,. \tag{2}$$

Here, $N(x)$ is the outward normal to D at $x \in \partial D$, and $dS(x)$ denotes integration with respect to surface area.

Read: Chapter 7.1-7.3 of Strauss.

Hand-in (due Friday, January 19):

(H1) Let D be a smooth bounded domain $\mathbb{R}^n$. Use the divergence theorem to show that the Neumann problem

$$\Delta u = f \text{ in } D\,, \quad \nabla u \cdot N = g \text{ on } \partial D$$

cannot have a solution unless $\int_D f \, dx = \int_{\partial D} g \, dS$.

(H2) Let D a connected bounded domain in $\mathbb{R}^n$. Prove that ...

(a) ... the Green's function is uniquely determined by its properties;

(b) ... $G(x,y) < 0$ for all $x, y \in D$ with $x \neq y$.

(H3) *(Strauss, Problem 7.2.2)* Let Φ be the fundamental solution of the Laplacian in $\mathbb{R}^n$, where $n \geq 3$. Given a bounded, continuous function f with compact support on $\mathbb{R}^n$, prove that

$$u(x) := \int_{\mathbb{R}^n} \Phi(x-y) f(y) \, dy$$

solves $\Delta u = f$ on $\mathbb{R}^n$.

(H4) *Weyl's lemma.* Let u be a continuous function on $\mathbb{R}^n$ that has the Mean Value Property. Let $H(x) = h(|x|$ be a smooth nonnegative radial function with compact support, with $\int_{\mathbb{R}^n} H(x) \, dx = 1$. (View H as the density of a radially symmetric probability measure.)

(a) Prove that

$$\int_{\mathbb{R}^n} H(x-y) u(y) \, dy = u(x) \quad \text{for all } x \in \mathbb{R}^n\,.$$

(b) Argue that u is therefore smooth. (Freely exchange derivatives with the integral.)

(c) Moreover, u is harmonic.

For discussion and practice: Strauss Problems 7.1.1, 7.1.2, 7.2.3, 7.3.2.