MAT 351: Partial Differential Equations
February 9, 2018

For the wave equation $u_{tt} = c^2\Delta u$, the heat equation $u_t = k\Delta u$, and the Schrödinger equation $iu_t = -\Delta u$, separation of variables leads to the same eigenvalue problem

$$-\Delta u = \lambda u\,.$$

It turns out that this eigenvalue problem has no solutions on $\mathbb{R}^n$ that decay at infinity or are even square integrable. (For every vector k, the function $u(x) = e^{-ik\cdot x}$ is a bounded solution with $\lambda = |k|^2$ but these don't lie in L^2.) So we had to investigate other methods of solutions.

- The solutions of the **wave equation** in one, two, and three spatial dimensions are given by the formulas of D'Alembert, Poisson and Kirchhoff. Similar formulas can be derived in higher dimensions.

- The solution of the **heat equation** with $u(x,u) = \phi(x)$ is given by

  $$u(x,t) = (4\pi kt)^{-n/2}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4kt}}\phi(y)\,dy\,.$$

  (The solution is not unique — the formula defines the solution that decays as $|x|\to\infty$, provided that ϕ itself decays.) The positivity of the heat kernel $(4\pi kt)^{-n/2}e^{-\frac{|x|^2}{4kt}}$ is a manifestation of the maximum principle.

- The solution formula for the heat equation remains valid, if k is a complex number with positive real part, provided that we take the square root $\sqrt{k}$ to have positive real part. The integral converges and defines a smooth function, so long as ϕ is bounded and integrable.

- By analytic continuation to $k = i$, we obtain for the **Schrödinger equation** the solution formula

  $$u(x,t) = (4\pi it)^{-n/2}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4it}}\phi(y)\,dy\,.$$

  Here, the square root in the first factor should be chosen as $\sqrt{i} = \frac{1+i}{\sqrt{2}}$. Note that the integral is now oscillatory, and will diverge unless ϕ itself decays at infinity. This is related to the wave-like properties of Schrödinger's equation. The fact that the kernel $(2\pi it)^{-n/2}e^{-\frac{|x-y|^2}{4it}}$ never vanishes indicates infinite speed of propagation.

All the formulas above are for solutions that live on the entire space, $\mathbb{R}^n$. We now return to the study of these equations on finite domains, with suitable boundary conditions, and revisit the Separation of Variables technique, separating the radial from the angular variables in polar coordinates. The resulting equations for the radial dependence give rise to special functions, such as Bessel functions. The angular part will be solved by spherical harmonics.

Read: The rmaining part of Chapter 9. Then move on into Chapter 10.

Assignment 13 (due Friday, February 16):

(H1) The solution of the wave equation $u_{tt} = c^2 \Delta u$ in space dimension $n = 2$ is given by Poisson's formula. Starting from this formula, use Hadamard's method of descent to recover D'Alembert's formula for the solution of the wave equation in space dimension $n = 1$.

(H2) *(Strauss, Problem 9.4.2)* Suppose that γ satisfies the **eikonal equation**

$$|\nabla \gamma(x)| = \frac{1}{c}, \qquad \text{for } x \in \mathbb{R}^n .$$

(a) Differentiate the equation to show that $\sum_{j=1}^{n} \gamma_{x_i x_j} \gamma_{x_j} = 0$ for each $i = 1, \ldots, n$.

(b) Let $x(t)$ be a solution of the differential equation $\dot{x} = c^2 \nabla \gamma(x)$. Show that $\ddot{x} = 0$, and hence $x(t)$ is a ray.

(c) Moreover, $h(x,t) = t - \gamma(x)$ is constant along this ray.

(d) Conclude that $S = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}^n \big| \, t - \gamma(x) = 0\}$ is a characteristic surface for the wave equation $u_{tt} = c^2 \Delta u$.

(H3) Let u be a solution of the two-dimensional wave equation with initial data supported on a disk $B_R(0)$.

(a) Prove that $tu(x,t)$ is bounded in t for fixed $x \in \mathbb{R}^2$, that is,

$$u(x,t) = O(t^{-1}) \quad \text{as } t \to \infty \, textforeach \, x \in \mathbb{R}^2 .$$

(b) Also prove that $t^{1/2} u(\cdot, t)$ is bounded *uniformly* in x as $t \to \infty$, that is,

$$\sup_{x \in \mathbb{R}^2} |u(x,t)| = O(t^{-1/2}) \quad \text{as } t \to \infty .$$

(H4) Derive the conservation of energy for the wave equation on a domain D with Dirichlet or Neumann boundary conditions. What about the Robin condition?

For discussion and practice:

1. (a) Solve the heat equation

$$u_t = k \Delta u \, , \quad u(x,0) = \phi(x)$$

for $x \in \mathbb{R}^n$ and $t > 0$ in the case where the initial values are given by a product of continuous functions with compact support

$$\phi(x) = \prod_{i=1}^{n} \phi_i(x_i) .$$

Assume that the solution is a product, $u(x,t) = \prod_{i=1}^{n} u(x_i, t)$. Use the solution in one dimension that we have constructed previously. Simplify your formula by combining the integrals and applying the rules of exponentiation.

(b) Argue that the formula holds in fact for *every* continuous function ϕ on $\mathbb{R}^n$ with compact support.