MAT 351: Partial Differential Equations
March 12, 2018

Consider the eigenvalue problem for the Laplacian on a domain $D \subset \mathbb{R}^d$ with Dirichlet boundary conditions

$$-\Delta u = \lambda u \text{ on } D\,, \qquad u = 0 \text{ on } \partial D\,.$$

We assume that D is bounded and that its boundary is smooth (e.g., the domein could be defined by an inequality $D = \{x \in \mathbb{R}^d \mid g(x) > 0\}$, where g is a smooth function that satisfies the hypotheses of the Implicit Function Theorem at every point where $g(x) = 0$.) Our goal is to prove that there is an infinite sequence of positive eigenvalues $\lambda_1 < \lambda_2 \le \dots$, whose growth is governed by **Weyl's law:** $\lambda_n \sim (4\pi^2)\left(\frac{n}{\operatorname{Vol} D}\right)^{\frac{2}{d}}$. Furthermore, we have completeness, i.e., $L^2(\mathbb{R}^d)$ has an orthonormal basis consisting of the corresponding eigenvectors $\{v_n\}$.

The main tool for the proof is the **variational characterization of eigenvalues:**

- **max-min:** $\lambda_n = \max\limits_{w_1,\dots w_{n-1}} \left\{ \min\limits_{u \perp w_1,\dots,w_{n-1}} \dfrac{\int_D |\nabla u|^2\,dx}{|u|^2} \right\}$;
- **min-max:** $\lambda_n = \min\limits_{w_1,\dots,w_n} \left\{ \max\limits_{u \in \operatorname{span}\{w_1,\dots,w_n\}} \dfrac{\int_D |\nabla u|^2\,dx}{||u||^2} \right\}$.

In these variational problems, the eigenvalues play the role of Lagrange multipliers. The objective functions is called the **Rayleigh quotient**. It is minimized by the lowest eigenvalue

$$\lambda_1 = \min_{||u||=1} \int_D |\nabla u|^2\,dx\,.$$

In these formulas, it is understood that $w_1, \dots, w_n$ and u should all satisfy the Dirichlet boundary conditions. For both the max-min and the min-max principle, the functions w_i must be linearly independent (but they need not be orthonormal). The min-max principle is widely used to obtain upper bounds on eigenvalues. The max-min principle can provide lower bounds, but it is difficult to apply, since it requires to solve two infinite-dimensional problems. The following finite-dimensional approximation method is surprisingly powerful.

- **Rayleigh-Ritz principle:** Choose n orthonormal *"trial functions"* $w_1, \dots w_n$ that satisfy the Dirichlet boundary conditions. Define a symmetric matrix A by

  $$A_{ij} = \int_D \nabla w_i \cdot \nabla w_j\,dx\,,$$

  and let $\mu_1 \le \dots \le \mu_n$ be its eigenvalues. Then $\lambda_i \le \mu_i$ for each $i = 1, \dots, n$.

(There are more complicated versions of this that do not require orthogonality.)

The proof of Weyl's law proceeds by comparing D with a finite union of rectangles. Once we have Weyl's law, we will obtain completeness of the eigenfunctions from the min-max principle.

Read: Sections 11.1 - 11.3.

Hand-in (due March 23):

(H1) Let $f(x)$ be a function on the interval $[0, 3]$ such that

$$f(0) = f(3) = 0\,, \quad \int_0^3 |f(x)|^2\,dx = 1\,, \quad \int_0^3 |f'(x)|^2\,dx = 1\,.$$

Find such a function if you can. If it cannot be found, explain why not.

(H2) Estimate the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions in the triangle

$$D = \{(x, y) \mid x + y < 1, x > 0, y > 0\}\,,$$

using the Rayleigh quotient with trial function $xy(1 - x - y)$.

(H3) Let D be a smooth, bounded domain in $\mathbb{R}^d$.

(a) Show that the smallest Neumann eigenvalue for $-\Delta$ on D is given by $\mu_0 = 0$. What is the corresponding eigenfunction?

(b) If, moreover D is connected, show that μ_0 is simple, by arguing that the next eigenvalue μ_1 is strictly positive. (You may use, without proof, that a smooth eigenfunction exists.)

(H4) Let D be a smooth bounded domain in $\mathbb{R}^d$. Denote by $(\lambda_n)_{n\geq 1}$ the sequence of eigenvalues of the negative Dirichlet Laplacian $-\Delta$, and by $(\phi_n)_{n\geq 1}$ an orthonormal basis of corresponding eigenfunctions.

Assume that u solves the heat equation $u_t = \Delta u$ on D, with Dirichlet boundary conditions $u|_{\partial d} = 0$ and initial values $u(x, 0) = f(x)$. Express u in terms of the Dirichlet eigenvalues and eigenfunctions defined above.

For discussion and practice: Problems in Section 11.2 of Strauss.