MAT 351: Partial Differential Equations
Assignment 1 — Sept. 15, 2017

Summary

The general **first order linear PDE** in two variables has the form

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = d(x,y)\,.$$

Initial conditions are given by prescribing a curve $\Gamma(s) = (x_0(s), y_0(s), u_0(s))$. The objective is to find an **integral surface** for the PDE that contains the initial curve. The **method of characteristics** builds the integral surface from curves that emanate from the initial curve by solving a system of ODE, as follows:

- Determine the **characteristics** in the (x,y)-plane by solving

$$\frac{dx}{dt} = a(x,y)\,, \quad \frac{dy}{dt} = b(x,y)\,. \tag{1}$$

- Along the characteristics, solve

$$\frac{d}{dt}z + cz = d\,, \tag{2}$$

  where $c = c(x(t), y(t))$, $d = d(x(t), y(t))$. The curves $(x(t), y(t), z(t))$ are called the **characteristic curves** of the PDE.

- Denote by $(x(t,s), y(t,s), z(t,s))$ the characteristic curve that passes through the point $\Gamma(s)$ on the initial curve at $t = 0$. The solution of the PDE is **implicitly defined** by

$$u(x(t,s), y(t,s)) = z(t,s)\,.$$

  This is a parametric representation of the integral surface. The final step is to eliminate the parameters and solve for $u(x,y)$.

Note that the characteristic equations (1) can be nonlinear, even when the PDE is linear, and hence its solutions may not be defined globally. Even if the characteristic equations have global solutions, the final step (the elimination of the parameters s, t) may be problematic. The Inverse Function Theorem guarantees that we can solve for $u(x,y)$ in some neighborhood of the initial curve, provided that $\Gamma(s)$ intersects the characteristics **transversally**, in the sense that

$$\det \begin{pmatrix} a(x_0(s), y_0(s)) & x_0'(s) \\ b(x_0(s), y_0(s)) & y_0'(s) \end{pmatrix} \neq 0\,.$$

Here, the first column is tangent to the characteristic, and the second column is tangent to the initial curve. In that neighborhood, the initial-value problem is well-posed.

Read: Chapter 1 of Strauss.

Hand-in (due September 22, in tutorial):

(H1) *(Divergence theorem in the large)*
If F is a continuous vector field on $\mathbb{R}^3$ and $|F(x)| \leq (1+|x|^3)^{-1}$, prove that

$$\int_{\mathbb{R}^3} \nabla \cdot F \, dx = 0 \, .$$

Hint: Consider a large ball B_R and then take $R \to \infty$.

(H2) *(The semigroup of dilations on $\mathbb{R}^n$)*
Fix $\alpha, \beta \in \mathbb{R}$. Find the general solution of the transport problem

$$u_t + \alpha x \cdot \nabla u + \beta u = 0$$

for $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

Problems for discussion:

1. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic (complex-differentiable) function. Write $z = x + iy$ and $f(z) = u(x,y) + iv(x,y)$, and interpret f as a function from $\mathbb{R}^2$ to itself.

   (a) The Cauchy-Riemann differential equations say that $u_x = v_y$, $u_y = -v_x$. Accepting this as a fact, show that u satisfies Laplace's equation $\Delta u = u_{xx} + u_{yy} = 0$ (and likewise for v).

   (b) Conversely, assume that $u : \mathbb{R}^2 \to \mathbb{R}$ satisfies Laplace's equation. (We say that u is a **harmonic function**). Show that there exists a function v such that the Cauchy-Riemann differential equations hold. (v is called a **conjugate harmonic function** to u. The function $u + iv$ is holomorphic on $\mathbb{C}$.)

2. A **multi-index** is a vector $\alpha = (\alpha_1, \dots, \alpha_n)$ of nonnegative integers. Define

   - $|\alpha| = \alpha_1 + \cdots + \alpha_n$, the **order** of α,
   - the power $x^\alpha = x_1^{\alpha_1} \cdot \dots \cdot x_n^{\alpha_n}$,
   - the **factorial** $\alpha! = \alpha_1! \cdot \dots \cdot \alpha_n!$,
   - the **multinomial coefficient** $\binom{|\alpha|}{\alpha} = \frac{|\alpha|!}{\alpha!} = \frac{|\alpha|!}{\alpha_1! \cdot \dots \cdot \alpha_n!}$.

   (a) Show that $(x_1 + \cdots + x_n)^k = \displaystyle\sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} x^\alpha$. (*Hint:* Induction over either k or n).

   (b) Let F be a smooth function on $\mathbb{R}^d$. Use the chain rule to compute (inductively)

   $$\frac{d^k}{(dt)^k} f(a+tx)\Big|_{t=0} .$$

3. *(Problem 1.5 of Pinchover-Rubinstein)*
   Let $p : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Prove that the equation $u_t = p(u)u_x$ has a solution that satisfies the functional relation $u(x,t) = f(x + p(u)t)$, where f is a differentiable function. In particular, find such solutions for the following equations:

   (a) $u_t = k u_x$, where k is a constant;

   (b) $u_t = u u_x$;

   (c) $u_t = u \sin u u_x$.