MAT 351: Partial Differential Equations
Assignment 2 — September 22, 2017

Summary:

A **conservation law** on the real line is a first order PDE of the form

$$u_t + \partial_x A(u) = 0\,,$$

where $x \in \mathbb{R}$ and $t \in \mathbb{R}$. We think of x as a spatial variable, and t as time. The function u is interpreted as the density of a (one-dimensional) fluid, and A describes the flow speed as a function of density. In gas dynamics, A is usually increasing and convex, while in traffic modeling it is increasing and concave. The simplest example is **Burger's equation**, where $A(u) = \frac{1}{2}u^2$, resulting in

$$u_t + uu_x = 0\,.$$

The results we discuss are representative for scalar conservation laws in one dimension. Systems of conservation laws in higher spatial dimensions, which appear in fluid dynamics, pose greater challenges that are beyond the scope of this course.

Conservation laws are examples of **quasilinear** equations, that is, equations that are linear in the highest order derivatives, with coefficients that depend on the unknown function. Solutions can be constructed by solving the characteristic equations

$$\begin{aligned} \dot{x} &= a(z) \\ \dot{z} &= 0\,, \end{aligned}$$

where x, z are real variables, and $a(z) = A'(z)$ is the derivative of A. What makes conservation laws interesting is that singularities can develop after a finite time, even when the initial values are smooth. This motivates the following definition.

- A function $u = u(x,t)$ is called a **weak solution** of the conservation law, if

  $$\int_{\partial D} \begin{pmatrix} A(u) \\ u \end{pmatrix} \cdot n = 0$$

  for every smooth domain $D \subset \mathbb{R}^2$. Here, $n = n(x,t)$ is the outward unit normal to D at a boundary point (x,t).

The prototypical singularity of a weak solution is a **shock**, where the value of the solution jumps across a smooth curve $x = \gamma(t)$. The speed of the shock is determined by the

- **Rankine-Hugoniot condition:** $\gamma'(t) = \dfrac{A(u_\ell) - A(u_r)}{u_\ell - u_r}$ at every point $(\gamma(t), t)$ on the curve.

We can think of the characteristic ODE as an infinitesimal version of the Rankine-Hugoniot condition (where the strength of the shock is taken to zero). For Burger's equation, the shock speed is just the average of the characteristic speeds immediately to the left and right of the shock.

It turns out that weak solutions at a given initial-value problem are not unique. Uniqueness is restored by requiring additionally that the solution satisfy

- **Lax' entropy condition:** At a shock, $a(u_\ell) > \gamma'(t) > a(u_r)$.

This means that nearby characteristics should always run *into* the shock. Characteristics emanating from a shock are viewed as unphysical. Note that the entropy condition breaks the symmetry of the PDE under the change of variables $(x,t) \to (-x,-t)$, and introduces a preferred direction of time. This is reminiscent of the second law of thermodynamics, which says that entropy always increases with time. One consequence is that shocks travel forward in gas dynamics, but backward in traffic modeling. A weak solution that satifies Lax' condition is called an **entropy solution**.

We note in passing that the method of characteristics can be adapted to solve first order fully nonlinear equations locally, i.e., in a neighborhood of the initial data, provided that a certain transversality condition holds. Examples of fully nonlinear first order equations are the eikonal equation $|\nabla u| = 1$ (which describes characteristic surfaces for the wave equation), and the Hamilton-Jacobi equation $u_t + H(u, \nabla u) = 0$ (which appears in classical mechanics).

Read: Chapter 1 and Section 14.1 of Strauss.

Hand-in (due September 29, in tutorial):

(H1) Use the method of **characteristics** to solve

(a) $xu_x + yu_y = 2u$, $u(x,1) = \phi(x)$;

(b) $u_x + u_y = u^2$, $u(x,0) = \phi(x)$;

(c) $uu_x + u_y = 1$, $u(x,x) = \frac{1}{2}x$.

Which of these equations is nonlinear? semilinear? quasilinear? fully nonlinear?

(H2) Write down an explicit formula function u solving the partical differential equation

$$u_t + b \cdot \nabla u + cu = f(x,t)$$

with initial values $u(x,0) = g(x)$. Here, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$ are constants, and f and g are smooth real-valued functions.

Problems for discussion:

1. State the Inverse Function Theorem (for continuously differentiable vector-valued functions on $\mathbb{R}^n$), using the word "well-posed". Likewise the Implicit Function theorem.

2. Consider the conservation law
$$u_t + \partial_x A(u) = 0\,,$$
where A is a non-decreasing function on the real line with $A(0) = 0$.

   (a) If this is a model for traffic flow, with $u(x,t) \geq 0$ denoting the density of cars, how would you interpret $a(u(x,t))$? Argue that the function $A(u)$ should be concave.

   (b) Let A be as above, and let u be an entropy solution. Interpret the formation of shocks, the Rankine-Hugoniot condition, and Lax' entropy condition in this traffic model.