MAT 351: Partial Differential Equations
Assignment 3 — September 29, 2017

The general second-order linear equation in n variables has the form

$$\sum_{i,j=1}^{n} a_{ij}(x)u_{x^i x^j} + b(x)\cdot\nabla u(x) + c(x)u = f(x)\,, \tag{1}$$

where $A(x) = (a(x)_{ij})$ is a symmetric $n\times n$ matrix, $b(x)$ a given vector fields, and $c(x)$, $f(x)$ are real-valued functions.

The local properties of the solutions of Eq. (1) is largely determined by the **type** of the equation, i.e, the signature of A. Eq. (1) is

- **elliptic**, if the eigenvalues $\lambda_1(x),\dots,\lambda_n(x)$ are either all positive, or all negative;
- **hyperbolic**, if one eigenvalue is positive and the others negative (or vice versa);
- **parabolic**, if exactly one eigenvalue is zero, and the others are of one sign.

In two dimensions ($n = 2$), the type is given by the sign of the determinant: If $\det A > 0$ the equation is elliptic, and if $\det A < 0$ it is hyperbolic. Note that the type of an equation can change as x varies over the domain.

The **wave equation**

$$u_{tt} = c^2\Delta u$$

is the prototype of a **hyperbolic equation**. It is used to describe the propagation of vibrations in an elastic medium such as a string or a membrane, as well as the propagation of electromagnetic waves in vacuum. In many cases, it is an approximation to a nonlinear wave equation that is valid for small amplitudes. The parameter c is called the **wave speed**. The wave equation is invariant under **time reversal**: If $u(x,t)$ solves the wave equation, then so does then $u(x,-t)$.

In one dimension, the general solution of the wave equation has the form $u(x,t) = f(x-ct) + g(x+ct)$. It can be expressed in terms of its initial amplitude $\phi(x) = u(x,0)$ and initial velocity $\psi(x) = u_t(x,0)$ by **d'Alembert's formula**

$$u(x,t) = \frac{1}{2}\big(\phi(x+ct) + \phi(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)\,dy\,.$$

The lines $x - ct = const.$ and $x + ct = const.$ are called the **characteristics** of the equation. The region between the characteristics that emanate from a point (x_0, t_0) with $t < t_0$ is called the **domain of dependence**, and the corresponding region with $t > t_0$ is called the **domain of influence**; together, they form the (solid) **light cone**. D'Alembert's formula implies that waves have **finite speed of propagation**, i.e., no signal can travel faster than at speed c. This is closely related with the idea of causality.

An important feature of the wave equation is that **energy is conserved**:

$$\frac{d}{dt}\int \frac{1}{2}|u_t(x,t)|^2 + \frac{c^2}{2}|u_x|^2\,dx = 0$$

(assuming that the integral is finite). Here, the first term in the integrand represents kinetic energy, and the second term represents the potential energy of the wave. Conservation of energy is useful for proving that the initial-value problem is well-posed in a suitable space of square integrable functions.

Solutions of the wave equation can be oscillatory (like $u(x,t) = \cos(ct)\cos x$) or traveling waves (given by $u(x,t) = f(x \pm ct)$); in higher dimensions, we will see examples of focusing (wave packets that are initially far apart collide in a small area) and dispersion (a wave packet separates into pieces that run off in different directions).

Read: Sections 1.6, 2.1, and 2.3 of Strauss.

Hand-in (due October 6, in tutorial):

(H1) Use the graphical method to sketch the solution of **Burger's equation** $u_t + uu_x = 0$ with initial values

$$u(x,0) = \begin{cases} 1 & \text{if } x < -1 \\ 0 & \text{if } -1 < x < 0 \\ 2 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$$

that satisfies both the Rankine-Hugoniot jump condition and the entropy condition. Be sure to indicate the location of the shocks and rarefaction waves.

(H2) Find the regions in the xy-plane where the equation

$$(1+x)u_{xx} + 2xyu_{xy} - y^2 u_{yy} = 0$$

is elliptic, hyperbolic, or parabolic.

(H3) Solve $u_{xx} - 3u_{xt} - 4u_{tt}$ with initial values $u(x,0) = x^2$, $u_t(x,0) = e^x$.
Hint: Factor the differential operator into two first order operators, as we did for the wave equation.

Problems for discussion:

1. Consider the initial-value problem for the wave equation with initial amplitude $\phi(x) = 0$ and initial velocity $\psi(x) = 1$ for $|x| < a$ and $\psi(x) = 0$ for $|x| \geq a$. Sketch the profile of the solution $u(x,t)$ as a function of x for $t = j\frac{a}{2c}$, $j = 0, \dots, 5$.

2. Solve the wave equation $u_{tt} = c^2 u_{xx}$ for $x \in \mathbb{R}$ with initial values $u(x,0) = \log(1+x^2)$, $u_t(x,0) = 4 + x$.

3. Consider the wave equation $u_{tt} - c^2 u_{xx} = 0$ with initial values $u(x,0) = \phi(x)$, $u_t(x,0) = \psi(x)$. If both ϕ and ψ are odd in x, prove that u is odd in x.

4. The PDE for **damped string** is given by $u_{tt} - c^2 u_{xx} + ru_t = 0$, where $r > 0$ is a parameter related to friction. Let $u(x,t)$ be a solution of this equation for $-\infty < x < \infty$ (i.e., for an infinitely long string) and assume that $u(x,t) \to 0$ as $x \to \pm\infty$. What is the energy for this equation? Prove that energy decreases with time.