MAT 351: Partial Differential Equations
Assignment 4 — October 6, 2017

The **diffusion equation**

$$u_t = k\Delta u$$

is the prototype of a **parabolic** equation. It is used to describe the diffusion of a chemical substance by Brownian motion, or the flow of heat in a body. A variant of this equation appears in the Black-Scholes equation for the price of a stock option. The parameter $k > 0$ is called the **diffusion constant** or the **volatility**. The diffusion equation is not time reversible; we will see that the initial-value problem is well-posed forward in time, but the backwards heat equation is ill-posed in most commonly used function spaces.

The most striking property of the heat equation is the **maximum principle**: If we consider a solution on a region $x \in D$, $t_0 \leq t \leq T$, then its maximal value is assumed either at at the initial time ($t = t_0$), or at the boundary of D. The **strong maximum principle** says that the maximum *cannot* be asssumed at some point (x_1, t_1) with x_1 in the interior of the interval and $t_1 > t_0$ unless u is constant up to time t_1. (We have proved the maximum principle on one dimension but not the strong maximum principle.) One consequence is that the solution of the heat equation with nonnegative data remains nonnegative. In fact, unless the data are zero, the solution will immediately become positive everywhere — the diffusion equation has **infinite speed of propagation** !

Typical solutions of the diffusion equation on the real line spread out and decay over time. An example of this is the function $u(x,t) = (4\pi kt)^{-\frac{1}{2}} e^{-\frac{x^2}{4kt}}$. One manifestation of this is that **energy decreases**:

$$\frac{d}{dt}\int \frac{1}{2}u^2(x,t)\,dx \leq 0$$

(assuming that the integral is finite). This is useful for understanding well-posedness and analyzing the long-time behavior.

The **fundamental solution** of the diffusion equation $u_t - ku_{xx} = 0$ is given by the **source function**

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}\,.$$

Physically, this represents a solution of the diffusion equation where all of the diffusing substance is concentrated at $x = 0$. The total mass of the substance is given by

$$\int_{-\infty}^{\infty} S(x,t)\,dt = 1$$

for all $t > 0$. The concentration of the substance at time t is given by a Gaussian bell-shaped curve that spreads out more and more as time passes. With the help of the fundamental solution, we can write the solution of the initial-value problem

$$\begin{aligned} &u_t = ku_{xx}\,, \quad (x \in \mathbb{R}, t > 0) \\ &u(x,0) = \phi(x) \end{aligned}$$

as

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy\,.$$

The formula says that the concentration $u(x,t)$ of the substance at position x, time t is a weighted average of the concentration at time $t = 0$.

Read: Sections 2-3 and 2.4 of Strauss.

Hand-in (due Friday, October 13):

(H1) Consider the diffusion equation on $-1 < x < 1$ with **Robin boundary conditions**

$$u_x(-1,t) - au(-1,t) = 0 = u_x(1,t) + au(1,t)\,.$$

- (a) If $a > 0$, show that the energy $E(t) = \int_{-1}^{1} u^2(x,t)\,dx$ decreases.
- (b) If a is much smaller than zero, show by example that energy may increase or decrease. *Hint:* Try to find solutions of the form $u(x,t) = h(t)\cosh(bx)$.

(H2) Prove the **comparison principle** for the diffusion equation: If u and v are two solutions of $u_t = ku_{xx}$ for $0 < x < \ell$ and $t > 0$, and $u \le v$ initially ($t = 0$) and on the boundary ($x = 0, \ell$), then $u(x,t) \le v(x,y)$ for all t.

Problems for discussion and practice:

1. Given u be a smooth function on $\mathbb{R}^2$, define $v(r\theta) := u(r\cos\theta, r\sin\theta)$. Change variables to express $\Delta u = u_{xx} + u_{yy}$ in terms of v and its partial derivatives $v_r, v_\theta, v_{rr}, v_{r\theta}$, and $v_{\theta\theta}$.
2. Write down the solution of the diffusion equation on the real line with initial condition $\phi(x) = 1$ for $|x| < a$, and $\phi(x) = 0$ for $|x| \ge a$.
3. (a) Explicitly solve the diffusion equation $u_t = u_{xx}$ on $0 < x < 1$ with Dirichlet boundary conditions, and initial values $u(x,0) = \sin(\pi x)$. (Try $u(x,t) = h(t)\sin(\pi x)$.)
   (b) Verify the **strict maximum principle** in this case, by showing that $u(x,t) < 1$ for all $t > 0$ and all $0 < x < 1$.
   (c) Verify that $E(t) = \int_0^1 u^2(x,t)\,dx$ is a strictly decreasing function of time.
4. In one or two sentences, explain:
   (a) The wave equation has finite speed of propagation, but the diffusion equation does not.
   (b) Why can't there be a maximum principle for the wave equation? Use the solutions you contructed on the previous assignment.