MAT 351: Partial Differential Equations
Assignment 5 — October 13, 2017

This week, we continued our discussion of second-order equations in two variables of the form

$$au_{xx} + 2bu_{xy} + cu_{yy} = F(x, y, u, u_x, u_y)\,. \tag{1}$$

Here, the coefficients a, b, c are functions of x, y. Such an equation is called **semilinear** (it is linear if $F = cu + du_x + eu_y + f$, where the coefficients c, d, e, f are functions of x, y.)

Denote the coefficient matrix by $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. A **characteristic direction** $v(x, y)$ is defined by a non-trivial solution of the equation the equation

$$v \cdot A(x, y)v = 0\,.$$

In terms of the usual classification, if (**??**) is ...

- ...**elliptic**, that is, $\det A > 0$, then A is either positive or negative definite.
  No characteristic directions
- ...**parabolic**, that is, $\det A = 0$ but $A \neq 0$, then A is semidefinite, with one zero eigenvalue.
  One characteristic direction, given by the eigenvector for the zero eigenvalue
- ...**hyperbolic**, that is, $\det A < 0$, then the eigenvalues of A have opposite sign.
  Two characteristic directions, $v_\pm = |\lambda_1|^{-1/2}v_1 \pm |\lambda_2|^{-1/2}v_2$, where (λ_i, v_i) are the eigenvalues and eigenvectors of A.
  Warning: This discussion applies only in two dimensions

Unless A is constant, the characteristic directions $v(x, y)$ change from point to point (and the type of A can also change). A **curve** $\gamma(t) = (x(t), y(t))$ is called **characteristic**, if its tangent vector $\dot\gamma(t)$ is a characteristic direction, that is, if

$$\dot\gamma(t) \cdot A(\gamma(t))\dot\gamma(t) = 0$$

for all t. If (**??**) is hyperbolic in some domain, one can change variables to a coordinate system (ξ, η) where the characteristics are horizontal and vertical lines, and (**??**) takes the form $u_{\xi\eta} = 0$. On the boundary of such domain, the equation degenerates to a parabolic equation as the characteristic directions become linearly dependent.

Keep reading: Chapter 2 of Strauss.

Hand-in (due Friday, October 20):

(H1) Let $u(x,t)$ be a smooth solution of the wave equation $u_{tt} = u_{xx}$ (set $c = 1$). Define the *energy density* $e(x,t)$ and the *momentum density* $p(x,t)$ by

$$e(x,t) = \frac{1}{2}(u_t^2 + u_x^2)\,, \qquad p(x,t) = u_t u_x\,.$$

1. Verify that $e_t = p_x$ and $p_t = e_x$.
2. Conclude that e and p also satisfy the wave equation.

(H2) *(Distortionless spherical waves with attenuation)*
Let $u(|x|,t)$ be a smooth radial solution of the wave equation $u_{tt} = c^2 \Delta u$ for $x \in \mathbb{R}^n$, $t \in \mathbb{R}$. (That is, u depends only on $|x|$, not the direction of x.)

1. Show that u satisfies the equation

$$u_{tt} = c^2 \left(u_{rr} + \frac{n-1}{r} u_r \right)\,.$$

   Hint: Direct computation, using the chain rule.
   (Avoid transforming first into polar coordinates).

2. Construct solutions of the form

$$u(r,t) = \alpha(r) f(t - \beta(r))\,.$$

   (Here, $\beta(r)$ is called the **delay**, and $\alpha(r)$ the **attenuation**.)
   Show that such solutions can exist in dimension for $n = 1$ and $n = 3$.
   Hint: Derive an ODE for f from the PDE. Then set the coefficients of f, f', f'' equal to zero.

3. In one dimension, show that α must be constant.

Problems for discussion and practice:

1. If $u(x,t)$ satisfies the wave equation $u_{tt} = u_{xx}$, and $h, k \in \mathbb{R}$, prove the identity

$$u(x+h, t+k) + u(x-h, t-k) = u(x+k, t+h) + u(x-k, t-h)\,.$$

   Sketch the quadrilateral whose vertices appear as the arguments in this identity.