MAT 351: Partial Differential Equations
Assignment 7 — October 27, 2017

Separation of variables is a method for finding special solutions of a partial differential equation. Sometimes we can generate all solutions of the PDE from these special solutions, using the

- **Superposition principle:** If the PDE is linear and homogeneous, then any linear combination of solutions is again a solution.

If, moreover, its coefficients are constant, then translates and derivatives of solutions are again solutions; if the equation has additional symmetries (such as rotations and dilations), they can be used to generate yet more solutions.

To explain the method, consider the wave equation on an interval

$$u_{tt} = c^2 u_{xx}\,, \quad (0 < x < \ell)\,,$$

with Dirichlet boundary conditions $u(0,t) = u(\ell,t) = 0$. We seek solutions that can be written as a product

$$u(x,t) = X(x)T(t)$$

for some unknown functions X and T. Inserting this into the PDE and collecting terms, we see that

$$\frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} =: -\lambda\,,$$

and deduce that λ can depend neither on x nor on t. We obtain a system of two ODE:

$$-X''(x) = \lambda X(x)\,, \quad -T''(t) = c^2\lambda T(t)\,.$$

Enforcing the Dirichlet boundary conditions $X(0) = X(\ell) = 0$, we conclude that

$$X(x) = \sin\beta x\,, \quad \lambda = \beta^2\,, \quad T(t) = A\cos(c\beta t) + B\sin(c\beta t)\,,$$

where $\beta = n\pi/\ell$ for some integer $n \geq 1$, and A and B are constants. We have found the special solutions

$$u_n(x,t) = \cos\Big(\frac{n\pi ct}{\ell}\Big)\sin\Big(\frac{n\pi x}{\ell}\Big)\,, \quad v_n(x,t) = \sin\Big(\frac{n\pi ct}{\ell}\Big)\sin\Big(\frac{n\pi x}{\ell}\Big)\,.$$

Two questions remain:

- Can we construct the **general solution** of the wave equation from the u_n and v_n by superposition?
- How can we determine the coefficients in the superposition from **initial values**?

Read: Chapter 4 of Strauss.

Hand-in (due Friday, November 3):

(H1) Consider Laplace's equation in polar coordinates in the plane

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\,, \quad (r > 0, \theta \in \mathbb{R})\,,$$

with periodic boundary conditions in θ, i.e., $u(r\theta + 2\pi) = u(r,\theta)$ for all θ.

(a) Set $u(r,\theta) = f(r)g(\theta)$ and separate variables to obtain a pair of ODE for f and g.

(b) Solve these ODE to obtain special solutions for the PDE. (*Hint:* Try $f(r) = r^\alpha$.)

(H2) Suppose that ϕ is bounded and continuous everywhere except for a jump discontinuity at a, i.e., the right- and left-sided limits

$$\phi(a^+) = \lim_{x\to a^+}\phi(x) \quad \text{and} \quad \phi(a^-) = \lim_{x\to a^-}\phi(x)$$

exist. Let S be the fundamental solution of the heat equation $u_t = ku_{xx}$, and set

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy\,.$$

(a) Explain why u solves the heat equation for $x \in \mathbb{R}$, $t > 0$.

(b) Show that for every $x \in \mathbb{R}$,

$$\lim_{t\to 0} u(x,t) = \frac{1}{2}\Big\{\phi(x^+) + \phi(x^-)\Big\}\,.$$

Hint: Change variables and prove that

$$\lim_{t\to 0^+}\frac{1}{\sqrt{\pi}}\int_0^\infty e^{-z^2/4}\phi(\sqrt{4kt}z)\,dz = \frac{1}{2}\phi(0^+)\,.$$

For discussion and practice:

1. A metal rod $(0 < x < \ell)$ insulated along its sides but not its ends is initially at a constant temperature u_0. Suddenly both ends are plunged into a bath of temperature zero.
(a) Write the initial-value problem for the temperature.
(b) Use the formula

$$1 = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k+1}\sin\Big(\frac{2\pi(k+1)x}{\ell}\Big)\,, \quad (0 < x < \ell)$$

to represent the solution $u(x,t)$ as a series.

2. Apply separation of variables to the Schrödinger equation $iu_t = u_{xx}$ with Dirichlet boundary conditions on $0 < x < 1$. Here, i is the imaginary unit, satisfying $i^2 = -1$. You may find Euler's formula useful: $e^{a+ib} = e^a(\cos b + i\sin b)$.

3. Find all eigenvalues of the linear operator $Au = 2u_{xxxx}$ on the space of 2π-periodic functions.

4. Use separation of variables to find solutions of the diffusion problem $u_t = ku_{xx}$ with mixed boundary conditions $u(0,t) = u_x(\ell,t) = 0$.