MAT 351: Partial Differential Equations
Assignment 8 — November 3, 2017

Separation of variables for the wave, heat, and Schrödinger equation on a bounded interval (a, b) lead to eigenvalue problems for the differential operator $-\partial_x^2$. Under suitable boundary conditions, the solutions of this eigenvalue problem satisfy infinite-dimensional analogues of the spectral theorem for Hermitian matrices.

To explain the analogy, define an inner product on the space of complex-valued continuous functions on (a, b) by

$$\langle u, v\rangle := \int_a^b u(x)\,\bar{v}(x)\,dx\,.$$

If u, v are twice continuously differentiable, and satisfy Dirichlet (or Neumann, or periodic) conditions, then

$$\langle -u'', v\rangle = \int_a^b u'(x)\,\bar{v}'(x)\,dx = \langle u, -v''\rangle\,.$$

In particular, $\langle -u'', u\rangle$ defines a positive definite quadratic form. Therefore, all eigenvalues of $-\partial_x^2$ are real and positive, and all eigenfunctions for distinct eigenvalues are mutually orthogonal. (For Neumann or periodic boundary conditions, the quadratic form is only positive semidefinite.) One important question remains:

- Are there *'enough'* eigenfunctions to that we can represent arbitrary continuous functions as superpositions?

This is the question of **completeness** of the eigenfunctions. To address it, we will spend some time studying the norm associated with the inner product by $||u||_2 := \sqrt{\langle u, u\rangle}$ (after Reading Week).

Read: End of Chapter 4; start Chapter 5 of Strauss.

Hand-in (due Friday, November 17):

(H1) (a) Find a pair a pair of ODE for X and Y such that $u(x, y) = X(x)Y(y)$ solves the PDE

$$u\Delta u = |\nabla u|^2\,, \quad (x, y \in \mathbb{R})\,.$$

(Do not try to solve these equations).

(b) Why does the superposition principle fail for this equation? Please explain!

(H2) *(Tychonoff's example of non-uniqueness for the heat equation)*
Consider the function $g(t) = e^{-1/(2t^2)}$ for $t > 0$, and set $g(0) = 0$ for $t \leq 0$.

(a) Show that g is differentiable, with $g'(t) = t^{-3}g(t)$ for $t > 0$, and $g'(0) = 0$. Moreover, g' is bounded. (*Hint:* L'Hopital's rule is useful here.)

(b) Argue by induction that g is smooth and has bounded derivatives of any order that vanish for $t \leq 0$.

(c) Set

$$u(x,t) = \sum_{k=0}^{\infty} g^{(k)}(t) \frac{x^{2k}}{2k!}, \quad x, t \in \mathbb{R}.$$

Differentiating term by term, argue that u formally solves the heat equation.

It remains to verify that the series converges absolutely, locally uniformly in x, t. This is a bit involved, and you are not asked to do that. (See the book of F. John, Chapter 7, or Bruce Driver's lecture notes on the heat equation on the web.)

For discussion and practice:

1. (a) On the interval $[-1, 1]$, show that the function x is orthogonal to the constant functions.
   (b) Find a quadratic polynomial that is orthogonal to both 1 and x.
   (c) Find a cubic polynomial that is orthogonal to all quadratics.
   (These are the first three Legendre polynomials.)

2. Let ϕ be a 2π-periodic function with Fourier series $\phi(x) = \sum_n A_n e^{inx}$.
   (a) If ϕ is real-valued, show that $A_{-n} = \bar{A}_n$.
   (b) If, additionally, ϕ is even, what can you say about the Fourier coefficients? Use this to represent ϕ as a cosine series.
   (c) What if ϕ is odd?

3. (a) Find the Fourier sine series of the function $f(x) = x$ on $[0, \pi]$.
   (b) Apply Parseval's identity to compute $\sum_{n=1}^{\infty} \frac{1}{n^2}$.
   (c) Integrate the sine series term by term to obtain a Fourier cosine series for the function $\frac{1}{2}x^2$. Note that the constant of integration appears as the $n = 0$ term in the series.
   (d) Then by setting $x = 0$, find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$.

4. Let γ_n be a sequence of constants with $\lim_{n\to\infty} \gamma_n = \infty$. Define a sequence of functions on $[0, 1]$ by $f_n(x) = \gamma_n \sin(n\pi x)$ for $0 \leq x \leq \frac{1}{n}$, and $f(x) = 0$ otherwise.
   (a) Show that $f_n \to 0$ pointwise, but not uniformly.
   (b) If $\gamma_n = n^{1/3}$, prove that $f_n \to 0$ in L^2.
   (c) If $\gamma_n = n^{2/3}$, show that f_n does not converge in L^2.