MAT 351: Partial Differential Equations
Test 2, January 25 2017

(Four problems; 20 points each.)

1. (a) Let H be an (infinite-dimensional, separable) Hilbert space. State Bessel's inequality and Parseval's identity.

   (b) Express the function $f(x) = x$ on the interval $(-\pi, \pi)$ as a **Fourier series**,
   $$f(x) = \sum_{k \in \mathbb{Z}} a_k e^{ikx} .$$

   (c) Consider the function $F(x) = \frac{1}{2}x^2$ on $(-\pi, \pi)$. Use the fact that $F' = f$ to determine the Fourier coefficients of F, i.e., find $(b_k)_{k \in \mathbb{Z}}$ such that
   $$F(x) = \sum_{k \in \mathbb{Z}} b_k e^{ikx} .$$
   Please justify your computation.

   (d) In what sense do the Fourier series of f and F converge? Please discuss briefly how the two examples differ, and why.

2. Let D be a bounded open subset in $\mathbb{R}^3$, with smooth boundary.

   (a) Define the **Green's function** of the domain, $G(x, y)$.

   (b) Let f be a continuous function defined on ∂D. Consider the the harmonic function u with boundary values f, that is, let u solve
   $$\Delta u(x) = 0 \quad \text{on } D ,$$
   $$u\big|_{\partial D} = f .$$
   Write down an integral formula for u in terms of f and G.

   (c) Prove that the Green's function of D is unique (assuming it exists).

   (d) Prove that the Green's function is negative, $G(x, y) < 0$ for $x, y \in D$.

3. Let u be the solution of the one-dimensional **wave equation**

$$u_{tt} = c^2 u_{xx}\,, \quad (x \in \mathbb{R}, t > 0)$$

with initial values

$$u(x,0) = 1 \quad \text{for } -2 < x < -1\,, \qquad u_t(x,0) = 1 \quad \text{for } 1 < x < 2\,.$$

(a) Sketch the regions in the x, t-plane where u vanishes.

(b) Also sketch the initial values and a profile the solution $u(\cdot, t)$ at time $t = \frac{1}{2c}$.

Hint: Use d'Alembert's formula

$$u(x,t) = \frac{1}{2}\big(\phi(x+ct) + \phi(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy\,.$$

4. The **Ginzburg-Landau equation**

$$u_t = \Delta u + u - u^3\,, \quad \text{for } x \in D, t > 0$$

describes the motion of interfaces through diffusion. Here, D be a smooth bounded domain in $\mathbb{R}^n$. We will impose Neumann boundary conditions

$$\nabla u \cdot n|_{\partial D} = 0\,.$$

(a) Verify that the constant functions $u = 0, \pm 1$ solve the problem.
(We refer to these as **steady-states**).

(b) If u is a solution of this equation, prove that the energy

$$E(t) = \int_D \frac{1}{2}|\nabla u|^2 + \frac{1}{4}(1-u^2)^2\,dx$$

can only decrease with time. (Assume in your calculation that the energy is finite and freely differentiate under the integral.)

(c) Show that the energy assumes its absolute minimum at the steady-states $u = \pm 1$.