APM 351: Differential Equations in Mathematical Physics
Test 3, March 16, 2012

(Five problems; 80 points total)

1. (10pts) How many linearly independent **harmonic polynomials** of order 6 are there in 3 variables (x, y, z)? Find the one that begins with $x^3y^3 + 0z + \dots$.

2. (20pts) Let $D \subset \mathbb{R}^d$ be a bounded domain with smooth boundary.

   (a) Write down the **Rayleigh principle** for the lowest eigenvalue of the Laplacian with Dirichlet boundary conditions.

   (b) Write down a corresponding (minimax or maximin) principle for the n-th eigenvalue λ_n.

   (c) If $D_1 \subset D_2$, how do the eigenvalues $\lambda_n(D_1)$ and $\lambda_n(D_2)$ compare? Please prove your claim!

   (d) If $\{\lambda_n\}$ are the Dirichlet eigenvalues of D, what are the Dirichlet eigenvalues of the scaled domain $\alpha D = \{\alpha x \mid x \in D\}$? How does this fit with **Weyl's law**?

3. (15pts) Consider the Dirichlet eigenvalue problem on the unit disc

$$-\Delta u = \lambda u \quad \text{for } x^2 + y^2 < 1\,, \qquad u\big|_{x^2+y^2=1} = 0\,.$$

   (a) Express the eigenvalues in terms of the **Bessel functions** J_n.

   (b) Prove that

$$\int_0^\infty J_n\big(z_{n,k}r\big)\,J_n\big(z_{n,\ell}r\big)\,r\,dr = 0$$

   for every $n \geq 0$ and all $k \neq \ell$. Here $\{z_{n,k}\}_{k\geq 1}$ are the zeroes of J_n.

4. (15pts) (a) Is $Y(\theta, \phi) = \cos\phi \sin\theta\,(\sin^2\theta - 4\cos^2\theta)$ a **spherical harmonic**? Of which order?

   (b) Solve the Poisson problem $\Delta u = 0$ on the unit ball $B = \{(x, y, z) : x^2 + y^2 + z^2 < 1\}$ with boundary values $u(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta) = Y(\theta, \phi)$.
   (You may use either spherical or Cartesian coordinates, but please explain!)

   (c) What is the solution of the corresponding Poisson problem on the exterior of the unit ball that decays at infinity?
   Hint: Try a negative power $r^{-\alpha}$.

5. (20pts) Consider a solution u of the **wave equation**

$$\begin{cases} u_{tt} = \Delta u\,, & x \in \mathbb{R}^3, t > 0 \\ u(x, 0) = 0, u_t(x, y) = \psi(x)\,, & x \in \mathbb{R}^3\,. \end{cases}$$

   Assume that ψ is smooth, bounded, and vanishes for $|x| > 1$.

   (a) Where does u have to vanish? (A sketch would be useful.)

   (b) Argue that $u(x, t) = O(t^{-1})$ *uniformly* as $t \to \infty$; that is, show that

$$\sup_{x,t}\{t \cdot |u(x, t)|\} < \infty\,.$$

   Hint: A spherical cap of radius r and opening angle α has area $2\pi r^2(1 - \cos\alpha) \leq Const.\, r^2\alpha^2$.

Useful formulas

- The **Laplacian in spherical coordinates** has the form

$$\Delta u = u_{rr} + \frac{2}{r} u_r + \frac{1}{r^2} \left[\frac{1}{\sin^2\theta} u_{\phi\phi} + \frac{1}{\sin\theta} (\sin\theta u_\theta)_\theta \right] .$$

  Here, $x = r\cos\phi\sin\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\theta$.

- The **Bessel functions** $J_n(r)$ are the (suitably normalized) bounded solutions of Bessel's equations

$$J'' + \frac{1}{r} J' + \left(1 - \frac{n^2}{r^2}\right) J = 0$$

  for $n = 0, 1, \dots$. Each J_n is a smooth function that changes sign at an infinite sequence of zeroes $z_{n,1}, z_{n,2}, \dots \to \infty$, separated by an infinite sequence of critical points $p_{n,1}, p_{n,2}, \dots$.

- **Kirchhoff's formula:** The solution of the three-dimensional wave equation $u_{tt} = \Delta u$ with speed of light $c = 1$ and initial values (ϕ, ψ) is given by

$$u(x_0, t_0) = \frac{\partial}{\partial t_0} \left[\frac{1}{4\pi t_0} \int_{|x-x_0|=t_0} \phi(x)\, dS(x) \right] + \frac{1}{4\pi t_0} \int_{|x-x_0|=t_0} \psi(x)\, dS(x) .$$