Knot Theory and Algebra

NSERC Research Proposal by Dror Bar-Natan
http://www.math.toronto.edu/~drorbn/Profile/ResearchProposal-07.pdf
last updated: Oct. 27, 2007

1. Algebraic Knot Theory

1.1. **Some general philosophy.** Let $\mathcal{K}$ be some algebraic structure. Namely, $\mathcal{K}$ is a set $\mathcal{K}_1$ or a number of sets $\mathcal{K}_i$, along with an operation α or a number of operations α_j defined on some of the sets or on their Cartesian products and taking values in one or another of the $\mathcal{K}_i$'s, possibly subject to some axioms. Examples include groups, vector spaces (two sets! {vectors} and {scalars}), categories, and in fact, almost everything we see in algebra.

A **structure-preserving invariant** Z on $\mathcal{K}$ is a morphism (in the obvious sense) from $\mathcal{K}$ into another algebraic structure $\mathcal{A}$ of the same kind as $\mathcal{K}$ (same numbers of sets and operations, same axioms). Invariants are useful, for example, when for some reason $\mathcal{A}$ is more "understandable" than $\mathcal{K}$ (perhaps the objects within $\mathcal{K}$ are knots or manifolds, and it is hard to tell if two are the same, while the objects within $\mathcal{A}$ are numbers or polynomials and are easier to work with).

Invariants in general are useful for telling things apart. Structure-preserving invariants are even more useful, for often they automatically detect (or exclude) "definable" properties, as follows.

A property P of elements of one of the sets $\mathcal{K}_i$ in $\mathcal{K}$ (or of a product of such $\mathcal{K}_i$'s) is called **definable** (within $\mathcal{K}$) if it is defined by a formula involving $\forall$, $\exists$, the logical "and", "or", "not", the operations α_j and perhaps a certain number of "constants" fixed in advance within the $\mathcal{K}_i$'s. Thus with some obvious operations, "being on the sphere" is definable within the reals, for $S^2 = \{(x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \colon x^2 + y^2 + z^2 = 1\}$.

If $Z : \mathcal{K} \to \mathcal{A}$ is a structure-preserving invariant and P is a definable property within $\mathcal{K}$, then the formula F defining P also defines a property P' of elements of $\mathcal{A}$. Depending on the precise order of quantifiers and logical operations in F it is often the case that for K in $\mathcal{K}$ the property $P(K)$ implies the property $P'(Z(K))$. If so, then we are in position to learn something about properties of K using the invariant Z and the presumably-easier target space $\mathcal{A}$.

This may sound lofty, yet it is precisely the principle underlying much of, say, **algebraic topology**. Indeed in algebraic topology we often prove the non-existence of some topological object (say a retract $r : D^2 \to S^1$) by applying an invariant Z (homotopy or homology, for example) that takes a hard category $\mathcal{K}$ (topological spaces) to an easier category $\mathcal{A}$ (groups) while preserving all relevant structure (the structure of a category). And indeed, a key point in the success of algebraic topology in saying something about retracts is the fact that the notion of a "retract" is definable within the given structure (in terms of a tiny commutative diagram or equivalently, a simple formula).

1.2. **The case of knots.** Knot invariants and especially "quantum" knot invariants are aplenty, and they are quite good at telling knots apart. But by en large, with a few notable exceptions, they fail at detecting properties of knots, such as their genus, unknotting number and (say) whether or not they are ribbon.

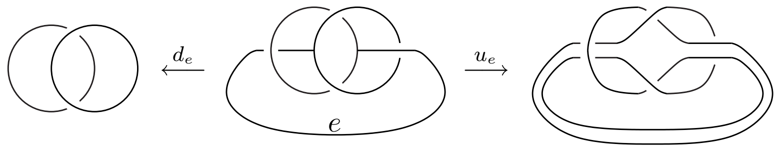


Figure 1: A KTG γ (middle), the result $d_e\gamma$ of deleting the edge e of γ (left) and the result $u_e\gamma$ of unzipping the edge e of γ (right). This figure also serves to suggest the definability of ribbon knots within knotted trivalent graphs: $u_e\gamma$ is clearly a ribbon knot while $d_e\gamma$ is clearly the unlink, and in general one sees that $\{u_e\gamma\colon \gamma \in \mathcal{K}(\circ\!\!\circ) \text{ and } d_e\gamma = \text{unlink}\} \subset \{\text{ribbon knots}\}$. The other inclusion requires more work and some modification of the statement.

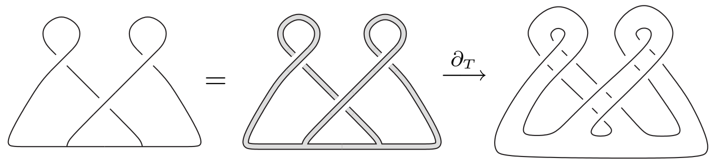


Figure 2: When we say "knotted graph", we really mean "knotted band graph". So a KTG γ is automatically a (thin) surface and its boundary $\partial_T(\gamma)$ is a knot of a pre-determined genus. It is possible to write the operation ∂_T as a composition of edge unzips and connect-sum operation with some simple constant KTGs, and hence knot genus is definable.

In my opinion, the underlying reason for that is that the language of knots and links is simply not expressive enough to render interesting properties definable. "Algebraic operations" on knots include the connected sum operation, cablings and a little more, but do not include even simple things such as crossing changes, for a "crossing change" is simply not a well defined operation — it also depends on further information, the choice of the crossing to be flipped. And it turns out that very little properties of interest can be stated just in terms of connected sums and cablings. Thus with what we have, we simply cannot expect there to be an **algebraic knot theory** with success comparable to that of algebraic topology.

The solution is to enlarge our domain space to also include knotted graphs, or more specifically, for a technical reason, **knotted trivalent graphs** (KTGs). Well-defined operations on KTGs include connected sum operations, edge deletion operations and unzip operations (see Figure 1) and with these operations, the genus, unknotting number and the set of ribbon knots (see e.g. Figures 1 and 2) all become definable.

2. My Proposed Research

It turns out that there is indeed an invariant Z on knotted trivalent graphs that is a morphism into some combinatorially defined structure, and which therefore may serve as a foundation for an algebraic knot theory. The invariant Z is a variant due to Murakami and Ohtsuki [MO] of the Kontsevich integral for knots [Ko, BN1]. Much of my research until now involved the study of the Kontsevich integral, in one way or another, though the "algebraic knot theory" perspective provides fresh motivation and new directions for study. The most general, of course, is the following:

Problem 1. Use the Kontsevich integral Z as an algebraic knot theory to obtain lower bounds on the genus of a knot and on unknotting numbers, to detect knots that are not ribbon (and may therefore be counterexamples to the $\{\text{ribbon}\} = \{\text{slice}\}$ conjecture), and in general, to say something about other KTG-definable classes of knots.

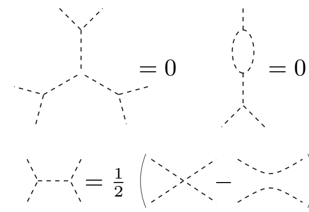


Figure 3: Two of the relations always satisfied by the weight system of the Alexander polynomial (top, [FKV]) and one of the relations likewise related to the Jones polynomial (bottom, folklore). Either set of relations reduces $\mathcal{A}$ to a manageable size (polynomial growth with reasonable generating sets); yet these reductions carry more than merely the Alexander and Jones polynomials (respectively), if only because these reductions make sense for knotted trivalent graphs rather than just for knots or links.

The Kontsevich integral Z is valued in some graded space of diagrams modulo relations, usually called $\mathcal{A}(\Gamma)$ where Γ is the trivalent graph being knotted. While in principle $\mathcal{A}(\Gamma)$ is combinatorially defined and computable to any given degree, it is still far from being understood. In particular, a solution of the following problem will give us a Kontsevich-integral method of bounding knot genus:

Problem 2. Determine $\operatorname{im} \partial_A$, where ∂_A is the operator defined on $\mathcal{A}$-spaces using the same composition that defines ∂_T in Figure 2.

Ng [Ng] has shown that no bounded-degree part of Z can be used to extract information about ribbon knots, though the full Z contains the Alexander polynomial, and hence contains at least some information on ribbon knots. A similar limitation applies to Z in relation to unknotting numbers. Thus to deal with ribbon knots and unknotting numbers we need an understanding of Z to all orders. It would be lovely to gain such an understanding, but unfortunately it seems beyond reach — it is implicit in [MO] that knowing $Z(\triangle)$ is equivalent to knowing a Drinfel'd associator (also see [Dr1, Dr2, LM, BN2]), and associators are notoriously difficult.

There is a potential way out, though. For the algebraic knot theory machinery to work and possibly be useful for ribbon knots and unknotting numbers, it is enough to consider quotients of $\mathcal{A}$ by appropriate ideals, such as the ones corresponding to the Alexander and Jones polynomials (Figure 3). In such quotients computing $Z(\triangle)$, or finding an associator, may well be within reach. (In some sense the Alexander quotient is already done, though it is not sufficiently understood. See Lieberum [Li]). Thus we come to our next two problems:

Problem 3. Develop a "theory of ideals", such as the ones generated by the relations in Figure 3. When are two such ideals "equivalent"? Can such ideals be classified? Bear in mind that in the similar algebraic world of ideals within polynomial algebras, ideals correspond to varieties and their study is the very rich subject called "algebraic geometry".

Problem 4. Which ideals lead to a computable theory? For those that do, compute and use your computations to study ribbon knots and unknotting numbers.

Algebraic knot theory suggests many other good problems, and the above problems can be sharpened and partitioned into many further problems ranging from topology via algebra to combinatorics, ranging from philosophical via concrete to computational and ranging from (probably) very hard to (probably) easy. Much of this is yet unwritten, but some of it can be found on my web site, especially at [BN4].

3. Knot Theoretic Algebra

I'll start with the question, and then try to put some meaning into it.

Question 1. Sometimes a bit of algebra turns out to be a bit of topology, in disguise. Is that true for the theory of quantum groups[1]?

3.1. **An example.** My favourite example for a bit of algebra which turns out to be a bit of topology is the Drinfel'd theory of associators[2].

It is well-known that associators can be used to produce knot and braid invariants (e.g. [LM, BN2]). But more is true. In [BN3] I have shown that a universal associator is *precisely* a certain type of braid invariant. Namely, universal associators are in a natural bijection with structure-preserving universal finite type invariants of a certain type of "parenthesized" braids, and hence an associator can equivalently be thought of as such an invariant. This equivalence is then used in [BN3] to better understand the algebra itself — within the equivalence lies a conceptual explanation for the appearance of a "Grothendieck-Teichmüller" group in Drinfel'd's study of associators, and the results of his study can be re-achieved in much simpler terms.

In fact, if knotted trivalent graphs are introduced as in Section 1.2 the picture becomes even nicer and the need for the topologically-artificial "parenthesizations" disappears. Indeed, one may show that a universal associator[3] is precisely the same as a structure-preserving universal finite type invariant of knotted trivalent graphs. Thus an associator is entirely a topological object, and one which arises rather naturally.

3.2. **What about quantum groups?** Can the same, or perhaps just nearly the same, be done to the theory of quasi-triangular Hopf algebras? The answer is a simple "no", if one wishes to stay within the world of ordinary knots. Finite type invariants of knotted objects are closely related to chord diagrams (modulo certain relations), and by the diagrammatic methods of [BN1], chord diagrams produce *invariant* elements of universal enveloping algebras of classical (non-quantum) Lie algebras. But when viewed non-quantumly, R-matrices, the fundamental building blocks of quantum groups, and are never invariant. Thus finite type invariants will never "see" the entire structure of a quantum group, and our program seems stuck.

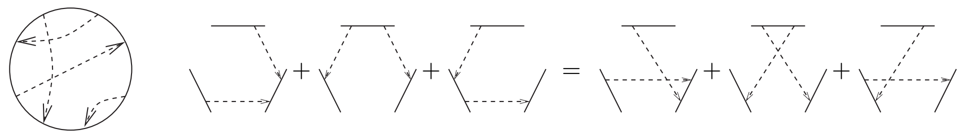
The way out is to consider "virtual knots", following Kauffman's [Ka]. Virtual knots are knots drawn on higher-genus surfaces, modulo stabilization by adding or removing empty handles. Ordinary knots inject into virtual knots [Ku] but there are many more virtual knots than ordinary knots (see a tabulation by my summer student J. Green, [Gr]).

The theory of finite type invariants of virtual knots is not well understood. But it is clear that much like finite type invariants of knots are related to chord diagrams modulo $4T$, finite type invariants of virtual knots are related to "arrow diagrams" modulo $6T$ (see Figure 4 and [Po]). Furthermore, "arrow diagrams" modulo $6T$ are the diagrammatic counterparts of Lie bialgebras (see my students' [Ha]), and Lie bialgebras are at the foundations of the theory of quantum groups — every Lie bialgebra can be naturally quantized ([EK] and subsequent papers by Etingof and Kazhdan), and the Drinfel'd-Jimbo quantization of any semi-simple Lie algebra arises from this construction. Finally, much of the work of Etingof and Kazhdan can be re-interpreted in diagrammatic terms, using arrow diagrams modulo $6T$ relations [Ha].

---

[1] Technically, "quasi-triangular power-series deformations of universal enveloping algebras".
[2] Technically, "quasi-Hopf power-series deformations of universal enveloping algebras".
[3] Technically, this time also allowing for non-horizontal chords.

Figure 4: An arrow diagram and the $6T$ relation.



Thus it is reasonable to expect that eventually a certain "universal quantum group" will arise, that should contain all other Drinfel'd-Jimbo quantum groups in much the same way as a universal associator can be projected to become an associator for any specific Lie algebra. And it is reasonable to expect that the universal quantum group will afford an interpretation as a structure-preserving universal finite type invariant of virtually-knotted objects, much like a universal associator is a structure-preserving universal finite type invariant of knotted trivalent graphs.

4. My Proposed Research

Over the grant period I plan, along with my students and collaborators, to make real the above expectations. Much remains to be done. Virtual knots are not well understood (there is a space of arrow diagrams but not yet a "Kontsevich theorem"), and virtual graphs are even less understood (exactly which graphs should we take, and with what operations?). And while much of the Etingof-Kazhdan work appears diagrammatic or even knot-theoretic, other parts still resist diagrammatization. In particular I don't understand the Etingof-Kazhdan crucial use of the PBW theorem, which screams "tangles", while for nearly the same objects and just a few lines apart, their use of "horizontal associators", which screams "braids".

5. Computations

My work is never very far from computations, and often when I see an algorithm, I implement it and post it on my web site along with the results of running it. I plan to keep things the same way. Everything in my proposal is in principle computable or will become so once sufficiently understood, and everything to come out of the research proposed here will be programmed, executed, documented and posted.

6. What About Khovanov Homology

I made significant contributions to the highly fashionable subject of Khovanov homology (in fact, while Khovanov is definitely the father of the field, I share the credit for making it fashionable...). Yet at the moment I don't feel mature enough to study this topic any further. I'd rather "categorify" knot invariants only after I properly understand the "algebra" on which they ought to be defined (in the sense of my first project above). And how can I even start categorifying other aspects of the theory of quantum groups, when in my opinion this theory in itself is poorly understood (at least in the sense of my second project)? With luck, at the end of this grant period I will be ready to return to Khovanov homology and categorification in general.

7. In Summary

My recent progress in research activities related to the proposal. I made significant contributions to almost every topic discussed in this proposal. See the "Contributions" section of my Form 100.

The objectives: both short and long term. To construct an "algebraic knot theory" in the spirit of "algebraic topology" and to show that quantum groups are a part of topology.
Literature pertinent to the proposal. See the "References" section of this proposal.
Methods and proposed approach. I plan to work both analytically, using the traditional mathematical definition-theorem-proof sequence, and by using computers for a large number of different computations.
Training to take place through the proposal. I expect that many parts of my proposed research will be assisted or carried out by graduate students and postdoctoral fellows as a part of their training.

References

[BN1] D. Bar-Natan, *On the Vassiliev knot invariants,* Topology **34** (1995) 423–472.

[BN2] D. Bar-Natan, *Non-associative tangles,* in *Geometric topology* (proceedings of the Georgia international topology conference), (W. H. Kazez, ed.), 139–183, Amer. Math. Soc. and International Press, Providence, 1997.

[BN3] D. Bar-Natan, *On Associators and the Grothendieck-Teichmuller Group I,* Selecta Mathematica, New Series **4** (1998) 183–212.

[BN4] D. Bar-Natan, *Algebraic Knot Theory — A Call for Action,* `http://katlas.math.toronto.edu/drorbn/index.php?title=AKT-CFA`.

[BN5] D. Bar-Natan, *Algebraic Knot Theory,* lecture notes and streaming video from a talk at Århus University, June 2007, `http://www.math.toronto.edu/~drorbn/Talks/Aarhus-0706`.

[Dr1] V. G. Drinfel'd, *Quasi-Hopf algebras,* Leningrad Math. J. **1** (1990) 1419–1457.

[Dr2] V. G. Drinfel'd, *On quasitriangular Quasi-Hopf algebras and a group closely connected with Gal*$(\bar{\mathbb{Q}}/\mathbb{Q})$*,* Leningrad Math. J. **2** (1991) 829–860.

[EK] P. Etingof and D. Kazhdan, *Quantization of Lie bialgebras, I,* Selecta Mathematica **NS 4(2)** (1998) 1022–1824, arXiv:q-alg/9506005.

[FKV] J. M. Figueroa-O'Farrill, T. Kimura and A. Vaintrob, *The universal Vassiliev invariant for the Lie superalgebra* $gl(1 \mid 1)$*,* Commun. Math. Phys. **185** (1997) 93–127, arXiv:q-alg/9602014.

[Gr] J. Green, *A Table of Virtual Knots,* `http://www.math.toronto.edu/~drorbn/Students/GreenJ`.

[Ha] A. Haviv, *Towards a diagrammatic analogue of the Reshetikhin-Turaev link invariants,* Hebrew University PhD thesis, September 2002, arXiv:math.QA/0211031.

[Ka] L. H. Kauffman, *Virtual Knot Theory,* European J. Comb. **20** (1999) 663–690, arXiv: math.GT/9811028.

[Ko] M. Kontsevich, *Vassiliev's knot invariants,* Advances in Soviet Mathematics, **16(2)** (1993) 137–150.

[Ku] G. Kuperberg, *What is a virtual link?,* Alg. Geom. Topol. **3** (2003) 587–591, arXiv:math.GT/0208039.

[LM] T. Q. T. Le and J. Murakami, *The universal Vassiliev-Kontsevich invariant for framed oriented links,* Compositio Math. **102** (1996), 41–64, arXiv:hep-th/9401016.

[Li] J. Lieberum, *The Drinfeld associator of* $gl(1|1)$*,* preprint, April 2002, arXiv:math.QA/0204346.

[MO] J. Murakami and T. Ohtsuki, *Topological Quantum Field Theory for the Universal Quantum Invariant,* Communications in Mathematical Physics **188** (1997) 501–520.

[Ng] K. Y. Ng, *Groups of ribbon knots,* Topology **37** (1998) 441–458, arXiv:q-alg/9502017 (with an addendum at arXiv:math.GT/0310074).

[Po] M. Polyak, *On the algebra of arrow diagrams,* Lett. Math. Phys. **51** (2000) 275–291.