"Expansions, Lie algebras and Invariants" CRM, Montreal
July 18, 2019, 11:00-12:30

# "Gate double derivatives"

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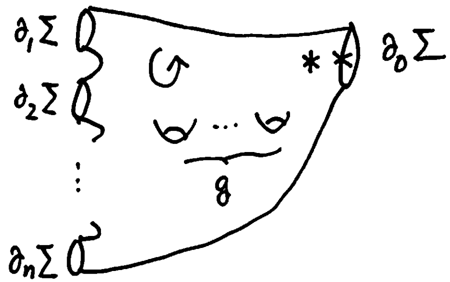
work in progress jointly with A. Alekseev, Y. Kuno and F. Naef

$\mathbb{K}$ : field of char. 0.

$\Sigma$ : compact connected oriented surface with $\partial\Sigma \neq \phi$

$\xRightarrow[\text{Classification}]{}$ $\exists g, \exists n \geq 0$, $\Sigma = \Sigma_{g,n+1} =$ 


$\chi(\Sigma_{g,n+1}) = 1-2g-n$

$* \in \partial_0\Sigma$ basepoint

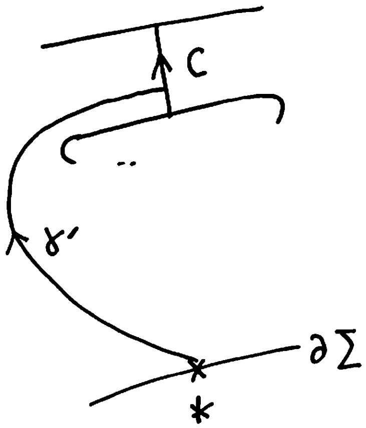
$\pi := \pi_1(\Sigma, *)$

free group of rank $2g+n$

$$\mathrm{aug} : \begin{array}{l} \mathbb{Z}\pi \to \mathbb{Z} \\ \mathbb{K}\pi \to \mathbb{K} \end{array} \qquad \sum_{\alpha\in\pi} a_\alpha \alpha \mapsto \sum a_\alpha \qquad \text{augmentation map.}$$

more general setting

Turaev (arXiv : 1901.02634), gate derivatives



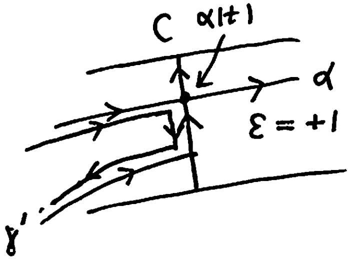

$C$ : gate, an oriented arc in $\Sigma$ connecting 2 distinct points on $\partial\Sigma$ away from the base point $*$

$\gamma' : ([0,1], 0, 1) \to (\Sigma, *, C)$ a path

$\partial_C^{\gamma'} : \mathbb{Z}\pi \to \mathbb{Z}\pi$ gate derivative

$\forall \alpha \in \pi$ choose its representative transverse to $C$

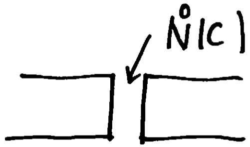
$$\partial_C^{\gamma'}(\alpha) \overset{\text{def}}{=} \sum_{t \in \alpha^{-1}(C)} \underbrace{\varepsilon_{\alpha(t)}(\alpha, C)}_{\text{local intersection}} \left(\alpha|_{[0,t]}\right) \cdot \begin{pmatrix} \text{subarc of } C \\ \text{from } \alpha(t) \\ \text{to } \gamma'(1) \end{pmatrix} \cdot (\gamma')^{-1} \in \mathbb{Z}\pi$$




Fox derivative $\forall a, \forall b \in \mathbb{Z}\pi$

$$\partial_C^{\gamma'}(ab) = \partial_C^{\gamma'}(a)\,\mathrm{aug}(b) + a\,\partial_C^{\gamma'}(b)$$

$\mathring{N}(C)$ : open tubular neighborhood

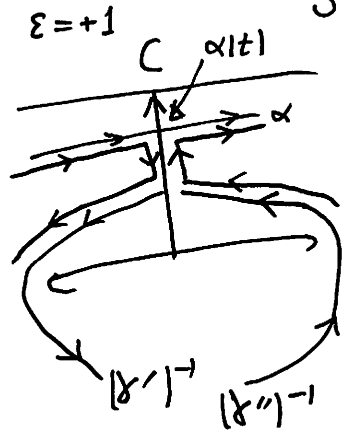



$$\chi(\Sigma \setminus \mathring{N}(C)) = \chi(\Sigma) + 1$$

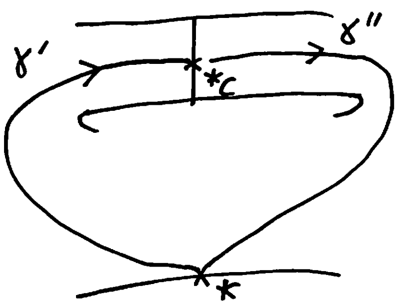
$(\because)$ $\{\Sigma \setminus C, \mathring{N}(C)\}$ open covering of $\Sigma$

$$\chi(C) = \chi(\Sigma \setminus C) + \chi(\mathring{N}(C)) - \chi(\mathring{N}(C) \setminus C) = \chi(\Sigma \setminus \mathring{N}(C)) + 1 - 2 \;/\!/$$

double version




## gate double derivatives




$*_C$ : a fixed mid point of $C$

$\gamma' : ([0,1], 0, 1) \to (\Sigma, *, *_C)$ paths

$\gamma'' : ([0,1], 0, 1) \to (\Sigma, *_C, *)$

s.t. $\gamma'([0,1[)$ and $\gamma''(]0,1])$ are disjoint from $C$

$\gamma'$ and $\gamma''$ are transverse to $C$

$\varepsilon_{*_C}(\gamma', C) = \varepsilon_{*_C}(\gamma'', C) = +1$

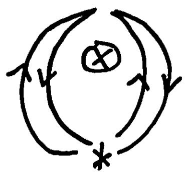
$\alpha \in \pi$ transverse to $C$

$$D(\alpha) \overset{\text{def}}{=} \sum_{t \in \alpha^{-1}(C)} \varepsilon_{\alpha(t)}(\alpha, C) \left(\alpha|_{[0,t]}\right) \begin{pmatrix}\text{subarc of } C \\ \text{from } \alpha(t) \text{ to } *_C\end{pmatrix} (\gamma')^{-1} \otimes (\gamma'')^{-1} \begin{pmatrix}\text{subarc of } C \\ \text{from } *_C \text{ to } \alpha(t)\end{pmatrix} \left(\alpha|_{[t,1]}\right) \in \mathbb{Z}\pi \otimes \mathbb{Z}\pi$$

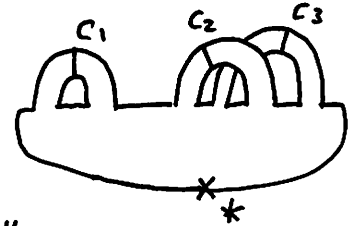
$D : \mathbb{Z}\pi \to \mathbb{Z}\pi \otimes \mathbb{Z}\pi$ gate double derivative associated to $C$, $\gamma'$ and $\gamma''$

double derivation $\forall a, \forall b \in \mathbb{Z}\pi \quad D(ab) = D(a)(1 \otimes b) + (a \otimes 1) D(b)$

$\gamma := \gamma' \gamma'' \in \pi$

$D(\gamma) =$  $= 1 \otimes 1 \in \mathbb{Z}\pi \otimes \mathbb{Z}\pi$

$D(\gamma^{-1}) = -\gamma^{-1} \otimes \gamma^{-1}$




## gate system

**Definition** $\mathcal{C} := \{C_i\}_{i=1}^{m}$ : a gate system on $\Sigma$

$\overset{\text{def}}{\Longleftrightarrow}$
0) $C_i$ : an oriented arc in $\Sigma$ connecting 2 distinct points on $\partial\Sigma$ away from $*$ $(1 \le i \le m)$ — a "gate"
1) $C_i \cap C_j = \phi$ if $i \neq j$
2) $\Sigma \setminus \coprod_{i=1}^{m} \mathring{N}(C_i) \cong D^2$ (2-dim. disk) $\longrightarrow$ $m = \chi(D^2) - \chi(\Sigma) = 2g+n$

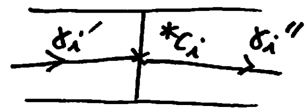
($\mathring{N}$: open tubular nbd)

$\overset{1:1}{\longleftrightarrow}$ (handle decomposition of $\Sigma$ with a single 0-handle and no 2-handles)

Choose $\gamma_i'$ and $\gamma_i''$ $(1 \le i \le m = 2g+n)$

s.t. $\gamma_i'([0,1[)$ and $\gamma_i''(]0,1])$ are disjoint from all $C_j$'s $(1 \le j \le 2g+n)$

$\gamma_i'$ and $\gamma_i''$ are transverse to $C_i$, $\varepsilon_{*_{C_i}}(\gamma_i', C_i) = \varepsilon_{*_{C_i}}(\gamma_i'', C_i) = +1$




- unique up to homotopy $(\because\ D^2 \simeq *)$
- $\gamma_i := \gamma_i' \gamma_i'' \in \pi$, $1 \le i \le m = 2g+n$

  $\{\gamma_i\}_{i=1}^{2g+n} \subset \pi$ free generators

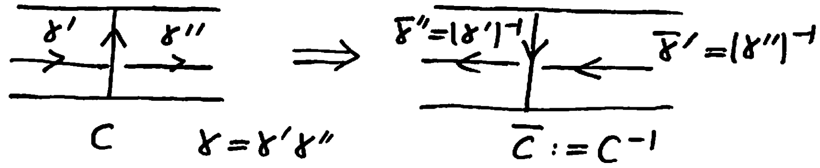
$$\frac{\partial}{\partial C_i} := \left(\begin{array}{l}\text{double gate derivative}\\ \text{associated with } C_i,\ \gamma_i' \text{ and } \gamma_i''\end{array}\right) : \mathbb{Z}\pi \to \mathbb{Z}\pi \otimes \mathbb{Z}\pi \quad \text{a double derivation}$$

$$\frac{\partial}{\partial C_i}(\gamma_j) = \delta_{ij}(1 \otimes 1) \qquad (1 \le \forall i, \forall j \le 2g+n)$$

## 2 kinds of elementary "coordinate changes"

① reversing the orientation of a gate

$$\frac{\partial\alpha}{\partial\overline{C}} = -(1\otimes\gamma)\frac{\partial\alpha}{\partial C}(\gamma\otimes 1)$$




(pf) drop "subarc of $C$ ⋯"

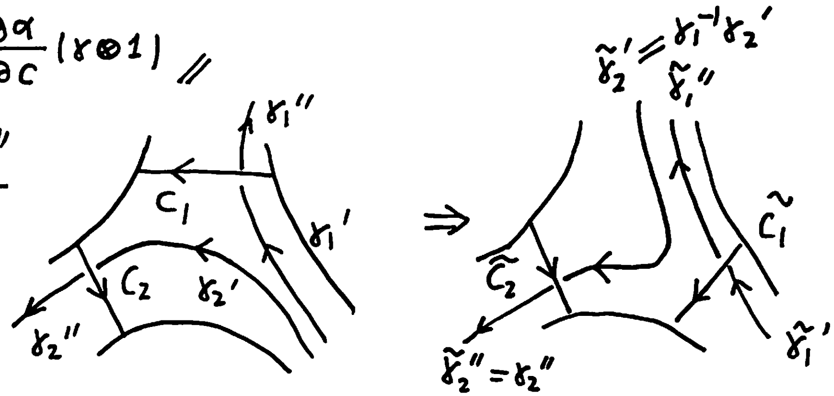
$$\frac{\partial\alpha}{\partial\overline{C}} = \sum_{t\in\alpha^{-1}(C)} \varepsilon_{\alpha(t)}(\alpha,\overline{C})(\alpha|_{[0,t]})\gamma''\otimes\gamma'(\alpha|_{[t,0]})$$

$$= -\sum_{t\in\alpha^{-1}(C)} \varepsilon_{\alpha(t)}(\alpha,C)(\alpha|_{[0,t]})(\gamma')^{-1}\gamma\otimes\gamma(\gamma'')^{-1}(\alpha|_{[t,0]})$$

$$= -(1\otimes\gamma)\frac{\partial\alpha}{\partial C}(\gamma\otimes 1) \quad //$$

② "elementary move"

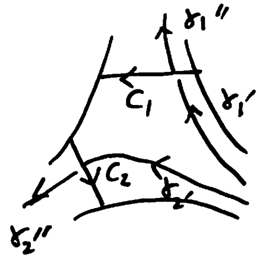
$$\begin{cases}\tilde{\gamma}_1 = \gamma_1\\ \tilde{\gamma}_2 = \gamma_1^{-1}\gamma_2\end{cases}$$

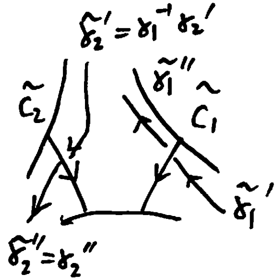



$$\begin{cases}\dfrac{\partial\alpha}{\partial\tilde{C}_1} = \dfrac{\partial\alpha}{\partial C_1} + (1\otimes\gamma_1^{-1}\gamma_2)\dfrac{\partial\alpha}{\partial C_2}\\[2ex] \dfrac{\partial\alpha}{\partial\tilde{C}_2} = \dfrac{\partial\alpha}{\partial C_2}(\gamma_1\otimes 1)\end{cases}$$

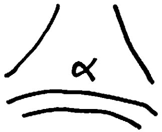
*proof* $$\frac{\partial\alpha}{\partial\widetilde{C}_2}=\sum_{t\in\alpha^{-1}(C_2)}\varepsilon_{\alpha(t)}(\alpha.C_2)\left(\alpha|_{[0,t]}\right)(\gamma_2')^{-1}\gamma_1\otimes(\gamma_2'')^{-1}\left(\alpha|_{[t,1]}\right)=\left(\frac{\partial\alpha}{\partial C_2}\right)(\gamma_1\otimes 1)$$

We check the formula on $\frac{\partial\alpha}{\partial\widetilde{C}_1}$ for 3 cases




Case 1




$$\frac{\partial\alpha}{\partial C_1}=0\ ,\quad \widetilde{\gamma}_2'=\gamma_1^{-1}\gamma_2'=\gamma_1^{-1}\gamma_2\,(\gamma_2'')^{-1}$$

$$\frac{\partial\alpha}{\partial\widetilde{C}_1}=\sum_{t\in\alpha^{-1}(C_2)}\varepsilon_{\alpha(t)}(\alpha.C_2)\left(\alpha|_{[0,t]}\right)(\gamma_2')^{-1}\otimes\widetilde{\gamma}_2'\left(\alpha|_{[t,1]}\right)$$

$$=\sum_{t\in\alpha^{-1}(C_2)}\varepsilon_{\alpha(t)}(\alpha.C_2)\left(\alpha|_{[0,t]}\right)(\gamma_2')^{-1}\otimes\gamma_1^{-1}\gamma_2(\gamma_2'')^{-1}\left(\alpha|_{[t,1]}\right)=(1\otimes\gamma_1^{-1}\gamma_2)\frac{\partial\alpha}{\partial C_2}\quad //_{\text{case }1}$$

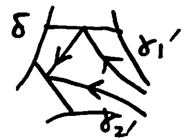
Case 2


$$\frac{\partial\alpha}{\partial C_2}=0$$

$$\frac{\partial\alpha}{\partial\widetilde{C}_1}=\sum_{t\in\alpha^{-1}(C_1)}\varepsilon_{\alpha(t)}(\alpha.C_1)\left(\alpha|_{[0,t]}\right)(\gamma_1')^{-1}\otimes(\gamma_1'')^{-1}\left(\alpha|_{[t,1]}\right)=\frac{\partial\alpha}{\partial C_1}\quad //_{\text{case }2}$$

Case 3


$\frac{\partial\alpha}{\partial\widetilde{C}_1}=0$, take a path $\delta$ from $*_{C_1}$ to $*_{C_2}$




$$\gamma_2'=\gamma_1'\delta$$
$$(\gamma_1'')^{-1}\delta=\widetilde{\gamma}_2'=\gamma_1^{-1}\gamma_2(\gamma_2'')^{-1}$$

$$\frac{\partial\alpha}{\partial C_1}=\sum_{t\in\alpha^{-1}(C_1)}\varepsilon_{\alpha(t)}(\alpha.C_1)\left(\alpha|_{[0,t]}\right)(\gamma_1')^{-1}\otimes(\gamma_1'')^{-1}\left(\alpha|_{[t,1]}\right)$$

$$=\sum_{t\in\alpha^{-1}(C_1)}\varepsilon_{\alpha(t)}(\alpha.C_1)\left(\alpha|_{[0,t]}\right)\delta(\gamma_2')^{-1}\otimes\gamma_1^{-1}\gamma_2(\gamma_2'')^{-1}\delta^{-1}\left(\alpha|_{[t,1]}\right)$$

$$=-\sum_{t\in\alpha^{-1}(C_2)}\varepsilon_{\alpha(t)}(\alpha.C_2)\left(\alpha|_{[0,t]}\right)(\gamma_2')^{-1}\otimes\gamma_1^{-1}\gamma_2(\gamma_2'')^{-1}\left(\alpha|_{[t,1]}\right)$$

$$=-(1\otimes\gamma_1^{-1}\gamma_2)\frac{\partial\alpha}{\partial C_2}$$

Hence $\frac{\partial\alpha}{\partial C_1}+(1\otimes\gamma_1^{-1}\gamma_2)\frac{\partial\alpha}{\partial C_2}=0=\frac{\partial\alpha}{\partial\widetilde{C}_1}$ $//_{\text{case }3}$ //

Notation Trace space (Lie algebra abelianization)

$\mathbb{K}$ : field of characteristic $0$

$A$ : (topological) associative $\mathbb{K}$-algebra $\ni 1$

$\rightsquigarrow$ $A$ : Lie algebra with $[a,b] := ab - ba$ $(a, b \in A)$

$|A| := A/[A,A]$ "trace space" (Lie algebra abelianization) $\mathbb{K}$-vector space

where $[A,A] :=$ (the closure of) the $\mathbb{K}$-linear span of $\{[a,b];\ a, b \in A\}$

$|\ | : A \to |A|$, $a \mapsto |a|$, quotient map

$\mathbb{1} := |1| \in |A|$.

$|A|' := |A|/\mathbb{K}\mathbb{1}$, $|\ |' : A \xrightarrow{|\cdot|} |A| \xrightarrow{\text{quotient}} |A|'$

ex 1) $\mathbb{K}\pi = \{\sum_{\alpha\in\pi} a_\alpha \alpha :\ a_\alpha \in \mathbb{K},\ a_\alpha = 0 \text{ except finite } \alpha\text{'s}\}$ group ring of $\pi = \pi_1(\Sigma, *)$

$|\mathbb{K}\pi| = \mathbb{K}(\pi/_{\text{conj}}) = \mathbb{K}[S^1, \Sigma]$ ← free homotopy set of free loops on $\Sigma$

$\alpha \in \pi \xmapsto{|\ |} |\alpha| \in [S^1, \Sigma]$ free homotopy class

similarly $|\mathbb{Z}\pi| = \mathbb{Z}[S^1, \Sigma]$, $|\mathbb{Z}\pi|' := |\mathbb{Z}\pi|/\mathbb{Z}\mathbb{1}$

ex 2) $V$ : fin dim. $\mathbb{K}$-vector space

$\hat{T}(V) := \prod_{m=0}^{\infty} V^{\otimes m}$ completed tensor algebra ← complete Hopf algebra (topology on $\hat{T}(V)$ ⇐ degree filtration)

$|\hat{T}(V)| = \prod_{m=0}^{\infty} (V^{\otimes m})_{\mathbb{Z}/m}$ cyclic coinvariants, $|\hat{T}(V)|' = \prod_{m=1}^{\infty} (V^{\otimes m})_{\mathbb{Z}/m}$

$\widehat{\mathbb{K}\pi}$ : completion of $\mathbb{K}\pi$ w.r.to a filtration $\{F_p\}_{p=0}^{\infty}$

top. isom $\cong$ $\varprojlim_{p\to\infty} \mathbb{K}\pi/(I\pi)^p$ , $I\pi := \mathrm{Ker}(\mathrm{aug})$ augmentation ideal

$$\log\gamma := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(\gamma-1)^n \in \widehat{\mathbb{K}\pi} \quad (\forall\gamma\in\pi)$$

$\frac{\partial}{\partial C_i} : \widehat{\mathbb{K}\pi} \to \widehat{\mathbb{K}\pi}\,\hat{\otimes}\,\widehat{\mathbb{K}\pi}$ (continuous) double derivation $(1\le\forall i\le m)$

<u>Lemma</u> $1\le\forall i\le m$. $\exists!\, \frac{\partial}{\partial\log\gamma_i} : \widehat{\mathbb{K}\pi} \to \widehat{\mathbb{K}\pi}\,\hat{\otimes}\,\widehat{\mathbb{K}\pi}$ (continuous) double derivation

s.t $\frac{\partial\log\gamma_j}{\partial\log\gamma_i} = \delta_{ij}\,(1\otimes1)$. $(1\le i,j\le m)$

(pf) (uniqueness) clear

(existence) $a\diamond b := a'b'\otimes b''a'' \in \widehat{\mathbb{K}\pi}\,\hat{\otimes}\,\widehat{\mathbb{K}\pi}$ , $a=a'\otimes a''$, $b=b'\otimes b''\in\widehat{\mathbb{K}\pi}\,\hat{\otimes}\,\widehat{\mathbb{K}\pi}$

$\forall w\in\widehat{\mathbb{K}\pi}$ $\frac{\partial w}{\partial\log\gamma_i} \overset{\text{def}}{=} \frac{\partial w}{\partial C_i} \diamond \frac{\gamma_i\otimes1-1\otimes\gamma_i}{(\log\gamma_i)\otimes1-1\otimes(\log\gamma_i)}$ //

<u>Lemma</u> $\forall u\in\mathrm{Der}(\widehat{\mathbb{K}\pi})$

$$\left|\frac{\partial u(\gamma_i)}{\partial C_i}\right| = \left|\frac{\partial u(\log\gamma_i)}{\partial\log\gamma_i}\right| + u_\circ\left(1\otimes|\log\gamma_i| + |r|\log\gamma_i\otimes1-1\otimes\log\gamma_i)|\right)$$

where $r(s) = \log\left(\frac{1}{s}(e^s-1)\right)$

## Turaev cobracket and its framed version

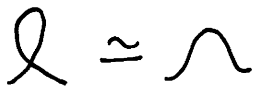
$\alpha \in \pi/\mathrm{conj} = [S^1, \Sigma]$ free loop, represented by a generic immersion (at worst transverse double pt.)

$D_\alpha := \{(t_1, t_2) \in S^1 \times S^1 \,:\, t_1 \neq t_2,\ \alpha(t_1) = \alpha(t_2)\}$

$$\delta\alpha := \sum_{(t_1,t_2)\in D_\alpha} \varepsilon(\dot\alpha(t_1), \dot\alpha(t_2))\, \big|\alpha|_{[t_1,t_2]}\big| \otimes \big|\alpha|_{[t_2,t_1]}\big| \in |\mathbb{Z}\pi| \otimes |\mathbb{Z}\pi|$$ Turaev cobracket


coskew

not well-defined $\Leftarrow$ monogon



original definition : take the quotient $|\mathbb{Z}\pi|' = |\mathbb{Z}\pi| / \mathbb{Z}\mathbb{1}$

$\delta : |\mathbb{Z}\pi|' \longrightarrow |\mathbb{Z}\pi|' \otimes |\mathbb{Z}\pi|'$ well-defined

$(|\mathbb{Z}\pi|', \text{Goldman bracket}, \delta)$ : Lie bialgebra (Turaev), involutive (Chas)

framed version fix a framing $T\Sigma \overset{f}{\cong} \Sigma \times \mathbb{R}^2$ $(\because \partial\Sigma \neq \emptyset)$ $\rightsquigarrow$ rotation number of immersed curves $\mathrm{rot}_f$

$\delta^f : |\mathbb{Z}\pi| \to |\mathbb{Z}\pi| \otimes |\mathbb{Z}\pi|$

$$\delta^f\alpha := \delta\alpha + (\mathrm{rot}_f\, \alpha)(\mathbb{1}\otimes\alpha - \alpha\otimes\mathbb{1}) \in |\mathbb{Z}\pi|$$

well-defined

<u>formal description of $\delta^f$</u>

$$H := H_1(\Sigma:\mathbb{K}) = H^{(1)} \supset H^{(2)} := \mathrm{Image}\,(H_1(\partial\Sigma:\mathbb{K}) \to H_1(\Sigma:\mathbb{K}))$$

$$\mathrm{gr}\, H = (H^{(1)}/H^{(2)}) \oplus H^{(2)}$$

$\exists \{F_p\}_{p=0}^{\infty}$ decreasing filtration on $\mathbb{K}\pi$

s.t. $$\mathrm{gr}(\mathbb{K}\pi) := \prod_{p=0}^{\infty} (F_p/F_{p+1}) \overset{\text{canon}}{\cong} \hat{T}(\mathrm{gr} H)$$

<u>Theorem</u> (Alekseev - K. - Kuno - Naef)

genus $(\Sigma) \neq 1$ or (genus 1 except some $f$)

$\Rightarrow$ $\exists\, \theta : \mathbb{K}\pi \to \hat{T}(\mathrm{gr} H)$ filtration preserving Hopf algebra homomorphism

completion w.r.to $\{F_p\}$

s.t. $\mathrm{gr}\,\theta = 1_{\hat{T}(\mathrm{gr} H)}$ ($\rightsquigarrow$ $\theta : \widehat{\mathbb{K}\pi} \xrightarrow{\cong} \hat{T}(\mathrm{gr} H)$ isom )

$|\theta| : (|\widehat{\mathbb{K}\pi}|$, Goldman bracket, $\delta^f) \xrightarrow{\cong} (\hat{T}(\mathrm{gr} H)$, $\mathrm{gr}$(Goldman bracket), $\mathrm{gr}(\delta^f))$

Lie bialgebra isom.

<u>Rmk 1</u> genus 0. Massuyeau using the Kontsevich integral

<u>Rmk 2</u>

Goldman bracket $\longleftrightarrow$ Kashiwara-Vergne problem (KV I)

Turaev cobracket $\longleftrightarrow$ (KV I) + (KV II)

<u>Rmk 3</u> (KV I) : special / symplectic condition

<u>Theorem</u> (K.-Kuno, Massuyeau-Turaev)

$\theta$ : (conjugate of) special / symplectic expansion

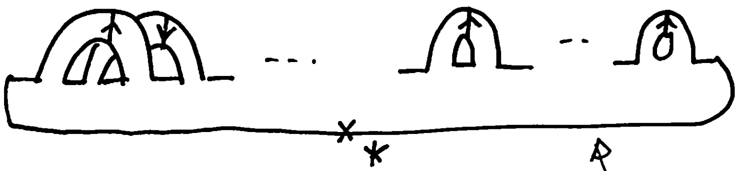
$\Rightarrow$ $|\theta|$ : Lie algebra isom.

<u>Theorem</u> (Alekseev-K.-Kuno-Naef)

$(\Leftarrow)$

## Key formula for the formal description of $\delta^f$

$A := \hat{T}(\mathrm{gr} H)$



$\Rightarrow \{\alpha_i, \beta_i, \gamma_j\} \subset \pi$ free generator

$x_i := [\alpha_i] \bmod H^{(2)}$, $y_i := [\beta_i] \bmod H^{(2)}$, $z_j := [\gamma_j]$ $\in \mathrm{gr} H$

$\theta_{\exp} : \mathbb{K}\pi \to A = \hat{T}(\mathrm{gr} H)$

$\alpha_i \mapsto e^{x_i}$, $\beta_i \mapsto e^{y_i}$, $\gamma_j \mapsto e^{z_j}$

"standard" gate system

$$\boxed{\delta^f_{\exp} = \underbrace{g\mathrm{Div}^f}_{(\mathrm{KV\,II})} \circ \underbrace{\hat{\sigma}_{\exp}}_{(\mathrm{KV\,I})}}$$

Alekseev-Torossian divergence

Massuyeau-Turaev

$$\delta^f_{exp} : (|\theta_{exp}| \otimes |\theta_{exp}|) \circ \delta^f \circ |\theta_{exp}|^{-1}$$

$$\hat{\sigma}_{exp} : (\theta_{exp})_* \circ \hat{\sigma} \circ |\theta_{exp}|^{-1} : |A| \to t\mathrm{Der}(A)$$

tangential

where $\hat{\sigma}$ : based version of the Goldman bracket (K.-Kuno).

$$g\mathrm{Div}^f : t\mathrm{Der}(A) \longrightarrow |A| \hat{\otimes} |A|$$

$\exists ! f_0 : T\Sigma \overset{\cong}{\Rightarrow} \Sigma \times \mathbb{R}^2$ framing

s.t. $\mathrm{rot}_{f_0} \alpha_i = \mathrm{rot}_{f_0} \beta_i = 0 \quad (1 \le {}^\forall i \le g)$, $\mathrm{rot}_{f_0}(\gamma_j) = 0 \quad (1 \le {}^\forall j \le n)$

$(\rightsquigarrow \mathrm{rot}_{f_0}(\partial_0 \Sigma) = \chi(\Sigma))$

$$g\mathrm{Div}^{f_0}(u) = \sum_{\substack{w = x_i, y_i, z_j \\ 1 \le i \le g \\ 1 \le j \le n}} \left| \frac{\partial}{\partial e^w} u(e^w) - 1 \otimes u(e^w) e^{-w} \right| \quad ({}^\forall u \in \mathrm{Der}(A))$$

where $\exists ! \dfrac{\partial}{\partial e^w} : A \longrightarrow A \hat{\otimes} A$ double derivation.

s.t. $\dfrac{\partial}{\partial e^w}(e^{w'}) = \delta_{w,w'}(1 \otimes 1) \quad ({}^\forall w, {}^\forall w' \in \{x_i, y_i, z_j\})$

( <u>Remark</u> If $u$ is tangential, then $|u(e^{z_j}) e^{-z_j}| = 0$ )

$g\mathrm{Div}^f$ was introduced by an <u>algebraic formula</u> in the context of non-commutative Poisson geometry

$\uparrow$ <u>topological interpretation</u> ?

$\Uparrow$

<u>gate double derivatives</u>

$\mathcal{C} = \{C_i\}_{i=1}^{2g+n}$ gate system

$\{\gamma_i\}_{i=1}^{2g+n} \subset \pi$ free generators associated with $\mathcal{C}$

$\mathrm{Div}^{\mathcal{C}} : \mathrm{Der}(\mathbb{Z}\pi) \to |\mathbb{Z}\pi| \otimes |\mathbb{Z}\pi|$

$u \in \mathrm{Der}(\mathbb{Z}\pi)$

$$\mathrm{Div}^{\mathcal{C}}(u) := \left| \sum_{i=1}^{2g+n} \frac{\partial}{\partial C_i} u(\gamma_i) - 1 \otimes u(\gamma_i)\gamma_i^{-1} \right| \in |\mathbb{Z}\pi| \otimes |\mathbb{Z}\pi|$$

( $\theta_{\exp *}(\mathrm{Div}^{\mathcal{C}}) = g\mathrm{Div}^{f_0}$ if $\mathcal{C}$ is "standard" )

"<u>Theorem</u>" $\mathrm{Div}^{\mathcal{C}}$ is independent of the choice of $\mathcal{C}$ up to framing terms

"<u>proof</u>" Compute the effects of two "coordinate changes"

① <u>reversing the orientation of a gate.</u>

$$\mathcal{C} \mapsto \overline{\mathcal{C}}$$
$$C \mapsto \overline{C} = C^{-1}$$
$$\gamma \mapsto \overline{\gamma} = \gamma^{-1}$$

$$u(\gamma^{-1}) = -\gamma^{-1}u(\gamma)\gamma^{-1}$$

$$\frac{\partial u(\gamma^{-1})}{\partial \overline{C}} = -\frac{\partial}{\partial \overline{C}}(\gamma^{-1}u(\gamma)\gamma^{-1}) = -1\otimes u(\gamma)\gamma^{-1} - (\gamma^{-1}\otimes 1)\left(\frac{\partial}{\partial \overline{C}}u(\gamma)\right)(1\otimes\gamma^{-1}) - \gamma^{-1}u(\gamma)\otimes 1$$

$$= -1\otimes u(\gamma)\gamma^{-1} - \gamma^{-1}u(\gamma)\otimes 1 + (\gamma^{-1}\otimes\gamma)\left(\frac{\partial}{\partial C}u(\gamma)\right)(\gamma\otimes\gamma^{-1})$$

$$\left|\frac{\partial u(\gamma^{-1})}{\partial \overline{C}} - \frac{\partial u(\gamma)}{\partial C}\right| = -\mathbb{1}\otimes|u(\gamma)\gamma^{-1}| - |u(\gamma)\gamma^{-1}|\otimes\mathbb{1}. \qquad (\mathbb{1} = |1|)$$

$$|-1\otimes u(\gamma^{-1})\gamma + 1\otimes u(\gamma)\gamma^{-1}| = 2\cdot\mathbb{1}\otimes|u(\gamma)\gamma^{-1}|$$

$$(\mathrm{Div}^{\overline{e}} - \mathrm{Div}^{e})(u) = \mathbb{1}\otimes|u(\gamma)\gamma^{-1}| - |u(\gamma)\gamma^{-1}|\otimes\mathbb{1}$$

$$= u\cdot(\mathbb{1}\wedge|\log\gamma|) \qquad \textit{framing term}$$

$$|\log\gamma| = [\gamma] \in H_1(\Sigma,\partial\Sigma:\mathbb{K}) \xlongequal{\text{Poincaré dual}} H^1(\Sigma:\mathbb{K})$$

$$\left( (\mathrm{Rmk})\ \ \mathrm{rot}_e\,\gamma = \mathrm{rot}_{\overline{e}}\,\gamma \quad (\because\ [\gamma]\cdot[\gamma] = 0) \right)$$

② elementary move $\mathcal{C} \mapsto \tilde{\mathcal{C}}$

$(C_1, C_2) \mapsto (\tilde{C}_1, \tilde{C}_2)$

$$\begin{cases} \hat{\gamma}_1 = \gamma_1 \\ \hat{\gamma}_2 = \gamma_1^{-1}\gamma_2 \end{cases} \qquad \begin{cases} \dfrac{\partial \alpha}{\partial \tilde{C}_1} = \dfrac{\partial \alpha}{\partial C_1} + (1 \otimes \gamma_1^{-1}\gamma_2) \dfrac{\partial \alpha}{\partial C_2} \\[2ex] \dfrac{\partial \alpha}{\partial \tilde{C}_2} = \dfrac{\partial \alpha}{\partial C_2} (\gamma_1 \otimes 1) \end{cases}$$

$$\begin{cases} u(\hat{\gamma}_1) = u(\gamma_1) \\ u(\hat{\gamma}_2) = u(\gamma_1^{-1}\gamma_2) = -\gamma_1^{-1} u(\gamma_1) \gamma_1^{-1}\gamma_2 + \gamma_1^{-1} u(\gamma_2) \end{cases}$$

$$\frac{\partial u(\hat{\gamma}_1)}{\partial \tilde{C}_1} - \frac{\partial u(\gamma_1)}{\partial C_1} = (1 \otimes \gamma_1^{-1}\gamma_2) \frac{\partial u(\gamma_1)}{\partial C_2}$$

$$\frac{\partial u(\hat{\gamma}_2)}{\partial \tilde{C}_2} = -\frac{\partial}{\partial C_2}(\gamma_1^{-1} u(\gamma_1) \gamma_1^{-1}\gamma_2)(\gamma_1 \otimes 1) + \frac{\partial}{\partial C_2}(\gamma_1^{-1} u(\gamma_2))(\gamma_1 \otimes 1)$$

$$= -(\gamma_1^{-1} \otimes 1) \frac{\partial u(\gamma_1)}{\partial C_2} (1 \otimes \gamma_1^{-1}\gamma_2)(\gamma_1 \otimes 1) - (\gamma_1^{-1} u(\gamma_1) \gamma_1^{-1} \otimes 1)(\gamma_1 \otimes 1)$$

$$+ (\gamma_1^{-1} \otimes 1) \frac{\partial u(\gamma_2)}{\partial C_2} (\gamma_1 \otimes 1)$$

$$\left| \frac{\partial u(\hat{\gamma}_2)}{\partial \tilde{C}_2} - \frac{\partial u(\gamma_2)}{\partial C_2} \right| = -\left| \frac{\partial u(\gamma_1)}{\partial C_1} (1 \otimes \gamma_1^{-1}\gamma_2) \right| - \left| \gamma_1^{-1} u(\gamma_1) \otimes 1 \right|$$

$$\left| \frac{\partial u(\hat{\gamma}_1)}{\partial \tilde{C}_1} + \frac{\partial u(\hat{\gamma}_2)}{\partial \tilde{C}_2} \right| - \left| \frac{\partial u(\gamma_1)}{\partial C_1} + \frac{\partial u(\gamma_2)}{\partial C_2} \right| = -\left| u(\gamma_1) \gamma_1^{-1} \right| \otimes \mathbb{1}$$

$$\begin{cases} -1\otimes u(\hat{\gamma}_1)\hat{\gamma}_1^{-1} + 1\otimes u(\gamma_1)\gamma_1^{-1} = 0 \\ |-1\otimes u(\hat{\gamma}_2)\hat{\gamma}_2^{-1} + 1\otimes u(\gamma_2)\gamma_2^{-1}| = |1\otimes \gamma_1^{-1}u(\gamma_1)\gamma_1^{-1}\gamma_2\gamma_2^{-1}\gamma_1| - |1\otimes\gamma_1^{-1}u(\gamma_2)\gamma_2^{-1}\gamma_1| + |1\otimes u(\gamma_2)\gamma_2^{-1}| \\ \qquad = \mathbb{1}\otimes|u(\gamma_1)\gamma_1^{-1}| \end{cases}$$

$$\mathrm{Div}^{\hat{\mathcal{E}}}(u) - \mathrm{Div}^{\mathcal{E}}(u) = \mathbb{1}\otimes|u(\gamma_1)\gamma_1^{-1}| - |u(\gamma_1)\gamma_1^{-1}|\otimes\mathbb{1}$$

$$= u.\,(\mathbb{1}\wedge|\log\gamma_1|)$$

framing term

// "Thm"