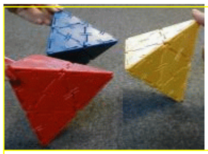
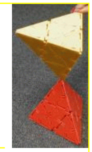


A 3-Dimensional Perspective on Drinfel'd's Theory of Quasi-Hopf Algebras

California Institute of Technology, February 21, 2000

Dror Bar-Natan, Hebrew University and MSRI



Quasi-Hopf algebra (Drinfel'd): (A, m, Δ, Φ) s.t. (typ. $A = \hat{U}(\mathfrak{g})$)

$$m : A \otimes A \to A, \quad \Delta : A \to A \otimes A, \quad \Phi \in A \otimes A \otimes A,$$

$$(I \otimes \Delta)(\Delta(a)) = \Phi \cdot (\Delta \otimes I)(\Delta(a)) \cdot \Phi^{-1}, \quad a \in A,$$

$$(I \otimes I \otimes \Delta)(\Phi) \cdot (\Delta \otimes I \otimes I)(\Phi) = (I \otimes \Phi) \cdot (I \otimes \Delta \otimes I)(\Phi) \cdot (\Phi \otimes I)$$

(↑ "the pentagon ⬠") (+more axioms...)

Why? It makes Rep(A) a tensor category; Δ defines $M_1 \otimes M_2$, Φ defines a map $(M_1 \otimes M_2) \otimes M_3 \to M_1 \otimes (M_2 \otimes M_3)$, and ⬠ ensures:

$$((M_1 \otimes M_2) \otimes M_3) \otimes M_4$$

$$\swarrow \qquad \searrow$$

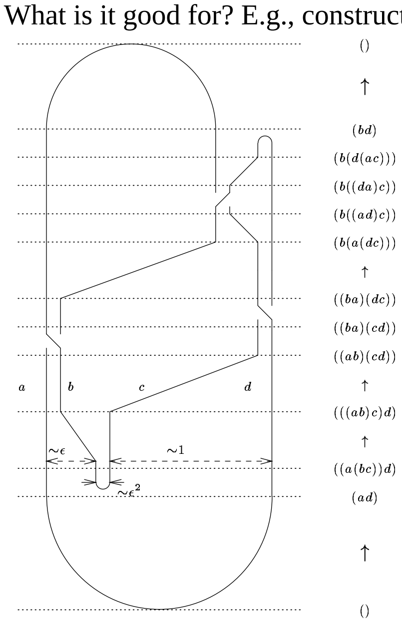
$$(M_1 \otimes M_2) \otimes (M_3 \otimes M_4) \qquad (M_1 \otimes (M_2 \otimes M_3)) \otimes M_4$$

$$\searrow \qquad \swarrow$$

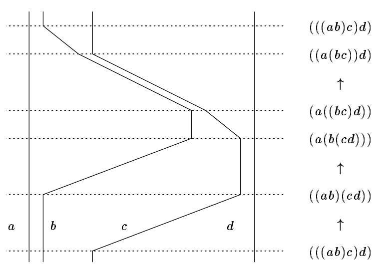
$$M_1 \otimes (M_2 \otimes (M_3 \otimes M_4)) \longleftarrow M_1 \otimes ((M_2 \otimes M_3) \otimes M_4)$$

Politically incorrect: this view is harmful to the categorically challenged!

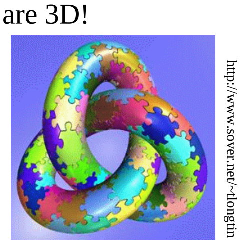
What is it good for? E.g., constructing link invariannts



⬠ becomes:



Why is it ugly? Braids are 1D; links are 3D!



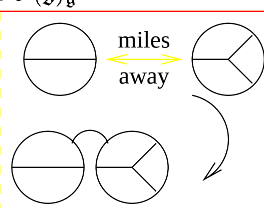
Chern-Simons-Witten Theory

$$Z(\gamma) = \int_{\mathfrak{g}\text{-connections}} \mathcal{D}A \, hol_\gamma(A) \exp\left[\frac{ik}{4\pi}\int_{\mathbb{R}^3} \operatorname{tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)\right]$$

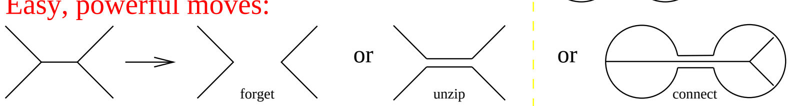
$$\text{ribbon link } \gamma \implies hol_\gamma(A) \text{ is } \begin{cases} \text{in } G \\ \text{dual to reps} \\ \text{in } \hat{U}(\mathfrak{g})_{\mathfrak{g}} \end{cases}$$

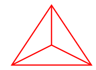
THE ONE THING TO REMEMBER:

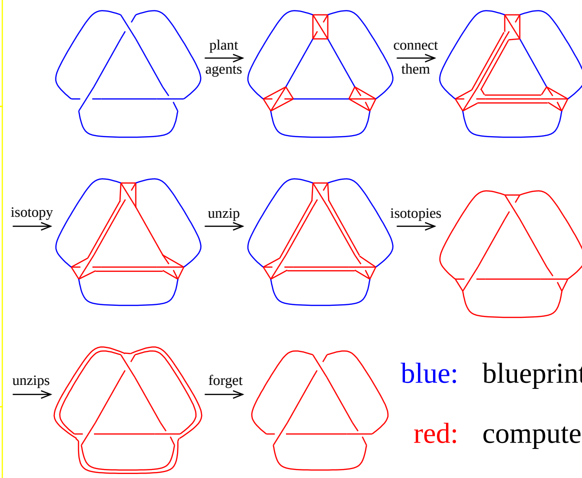
Embedded Trivalent ribbon Graph γ $\implies$ $hol_\gamma(A)$, $Z(\gamma)$ $\in$ $\hat{U}(\gamma) := \hat{U}(\mathfrak{g})^{\otimes E(\gamma)} / \mathfrak{g}^{V(\gamma)}$



Easy, powerful moves:



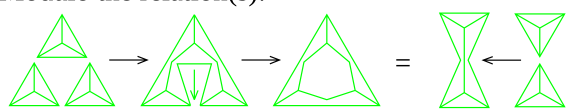
Using moves, ETG is generated by ribbon twists and the tetrahedron  :



blue: blueprint

red: computed

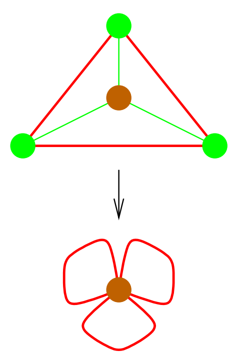
Modulo the relation(s):

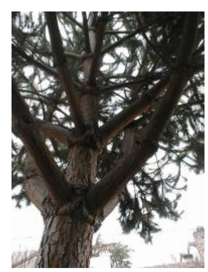


(+more)

Claim: $\hat{U}(\text{⚠}) \equiv \hat{U}(\mathfrak{g})^{\otimes 3}/\mathfrak{g}$ and under this isomorphism the above relation for $Z(\text{⚠})$ becomes the pentagon ⬠ for Φ, and likewise for all other Drinfel'd's axioms.

Proof: Collapse a tree:





constant valency tree
Scenic cross Cedar
Berkeley

Why am I happy?

1. 3-dimensional picture of associators (Φ's).
2. A direct link between CSW and quasi-Hopf.
3. Will have applications...

Joint with Dylan Thurston

This handout is at
http://www.ma.huji.ac.il/~drorbn/Talks/CalTech-000221