From the $ax+b$ Lie Algebra to the Alexander Polynomial and Beyond

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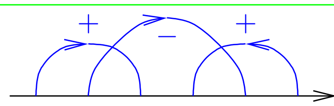
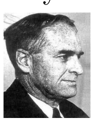
http://www.math.toronto.edu/~drorbn/Talks/Chicago-1009/

Abstract. I will present the simplest-ever "quantum" formula for the Alexander polynomial, using only the unique two dimensional non-commutative Lie algebra (the one associated with the "$ax+b$" Lie group). After introducing the "Euler technique" and some diagrammatic calculus I will sketch the proof of the said formula, and following that, I will present a long list of extensions, generalizations, and dreams.

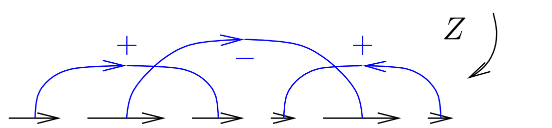
The 2D Lie Algebra. Let $\mathfrak{g}=\mathfrak{lie}(x^1,x^2)/[x^1,x^2]=x^2$, let $\mathfrak{g}^*=\langle\phi_1,\phi_2\rangle$ with $\phi_i(x^j)=\delta_i^j$, let $I\mathfrak{g}=\mathfrak{g}^*\rtimes\mathfrak{g}$ so $[\phi_i,\phi_j]=[\phi_1,x^i]=0$ while $[x^1,\phi_2]=-\phi_2$ and $[x^2,\phi_2]=\phi_1$. Let $r=Id=\phi_1\otimes x^1+\phi_2\otimes x^2\in\mathfrak{g}^*\otimes\mathfrak{g}\subset I\mathfrak{g}\otimes I\mathfrak{g}$. Let $\mathcal{U}=\{\text{words in } I\mathfrak{g}\}/ab-ba=[a,b]$, degree-completed with respect to $\deg\phi_i=1$ and $\deg x^i=0$ (so $\mathcal{U}\equiv$ (power series is 4 variables)). Let $R=\exp(r)\in\mathcal{U}\otimes\mathcal{U}$.

The Invariant. Define Z : {long knots} $\to\mathcal{U}$ by mapping every $\pm$-crossing to $R^{\pm1}$:





Alexander

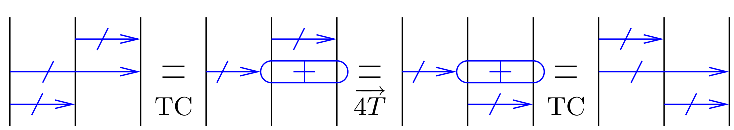


$$\cdots+\tfrac{1}{2!}\tfrac{(-1)^3}{3!}\tfrac{1}{1!}(\phi_2\phi_1)(\phi_2\phi_1\phi_2)(x^2x^1)(x^1)(x^2x^1x^2)(\phi_1)+\cdots$$

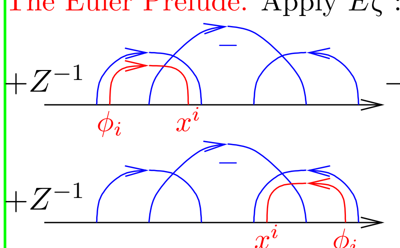
Near Theorem. Z is invariant, and it is essentially the Alexander polynomial; with $N=\exp(\overrightarrow{l}\,\phi_ix^i+\overleftarrow{l}\,x^i\phi_i)=:\exp(SL)$,

$$Z(K)=N\cdot\left(A(K)(e^{\phi_1})\right)^{-1}\qquad(1)$$

Invariance. "The identity is an invariant tensor":

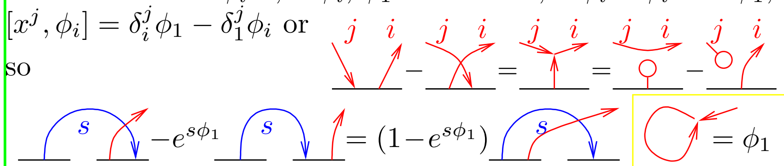


The Euler Prelude. Apply $\tilde{E}\zeta:=\zeta^{-1}E\zeta$ to (1):

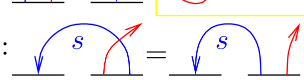


$$\stackrel{?}{=}SL-\phi_1\operatorname{tr}\left(M^{-1}\tfrac{d}{d\phi_1}M(e^{\phi_1})\right)$$

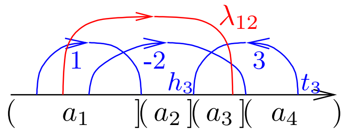
Some Relations. ϕ_ix^i, $x^i\phi_i$, ϕ_1 are central, $x^i\phi_i-\phi_ix^i=\phi_1$, $[x^j,\phi_i]=\delta_i^j\phi_1-\delta_1^j\phi_i$ or so



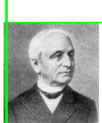
and the famed "tails commute" (TC):



Near Proof. Let $\lambda_{\alpha j}$ be a red arrow with tail at a_α and head just left of h_j. Let $\Lambda=(\lambda_{\alpha j})$. Then roughly $R\Lambda=\phi_1I$ so roughly, $\Lambda=R^{-1}\phi_1$. The rest is book-keeping that I haven't finished yet, yet with which my computer agrees fully.



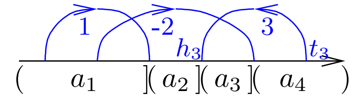
I don't understand the Alexander polynomial!



"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

www.katlas.org The Knot Atlas

An Alexander Reminder. Number the arrows $1,\ldots,n$, let t_j,h_j be the tail and head of arrow j, and let $s_j\in\pm1$ be its sign. Cut the skeleton into arcs a_α by arrow heads, and let $\alpha(p)$ be "the arc of point p". Let $R\in M_{n\times(n+1)}$ be the matrix whose j'th row has -1 in column $\alpha(h_j)$ and $1-X^{s_j}$ in column $\alpha(t_j)$ and X^{s_j} in column $\alpha(h_j)+1$, and let M be R with a column removed. Then $A(X)=\det(M)$.



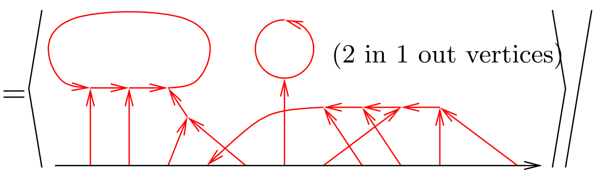
$$\begin{pmatrix}1-X+(-1) & X & 0 & 0\\ 1-X^{-1} & 0 & -1 & X^{-1}\\ 0 & -1 & X & 1-X\end{pmatrix}$$

An Euler Interlude. If you know brackets, how do you test exponentials? When's $e^Ae^B=e^Ce^D$?

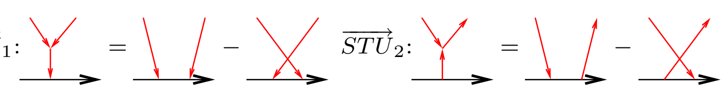
Bad Idea. Take log and use BCH. You'll want to cry.

Clever Idea. Let E be the Euler derivation, which multiplies each element by its degree (e.g. on $\mathbb{Q}[\![\phi]\!]$, $Ef=\phi\partial_\phi f$, so $Ee^\phi=\phi e^\phi$). Apply $\tilde{E}\zeta:=\zeta^{-1}E\zeta$: $\tilde{E}(e^Ae^B)=e^{-B}e^{-A}\left(e^AAe^B+e^Ae^BB\right)=e^{-B}Ae^B+B=e^{-\operatorname{ad}B}(A)+B$.

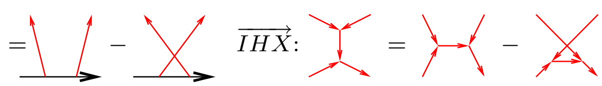
"Uninterpreting" Diagrams. Make $Z^w:\mathcal{K}^w\to\mathcal{A}^w\to\mathcal{U}$, with



$\mathcal{A}^w=\langle$ (2 in 1 out vertices) $\rangle$ / $\overrightarrow{STU}$, $\overrightarrow{AS}$, and $\overrightarrow{IHX}$ relations



$\overrightarrow{STU}_1$, $\overrightarrow{STU}_2$, $\overrightarrow{STU}_3$ =TC: 0 =, $\overrightarrow{IHX}$



$\mathcal{K}^w=CA\langle\ \rangle$ / R23, OC

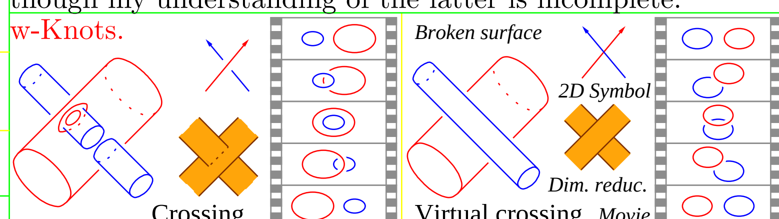
$=PA\langle\ \rangle$ / R23, VR123, D, OC

UC

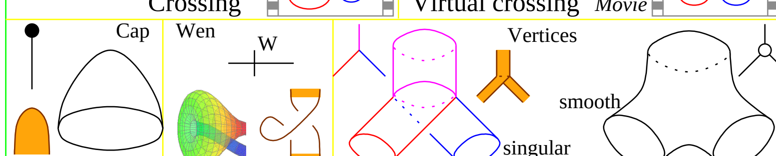
R3 VR3 D OC

Z^w is a UFTI on w-knots! It extends to links and tangles, is well behaved under compositions and cables, and remains computable for tangles. It contains Burau, Gassner, and Cimasoni-Turaev in natural ways, and it contains the MVA though my understanding of the latter is incomplete.

w-Knots.



Crossing; Broken surface; 2D Symbol; Dim. reduc.; Virtual crossing; Movie



Cap; Wen; W; Vertices; smooth; singular

There's 1D in 4D, non-trivial given 2D, and there are ops...

Dream. Z^w extends to virtual knots as $Z^v:\mathcal{K}^v\to\mathcal{A}^v$, with good composition and cabling properties and plenty of computable quotients, more then there are quantum groups and representations thereof. I don't understand quantum groups!