Dror Bar-Natan: Talks: Hanoi-0708: Following Lin:

Expansions for Groups

Riverside, April 2000

Kyoto, September 2001

Vaughan's Hierarchy (generalized, unauthorized)

- ☺ Computation
- ☺ Formula
- 😐 Proof
- ☹ Theory
- ☹ Dream

See Lin's "Power Series Expansions and Invariants of Links", 1993 Georgia International Topology Conference, AMS/IP Studies in Adv. Math. **2** (1997) 184–202.

The Magnus and Exponential Expansions

$$Z_{1,2}: G_n = \begin{pmatrix}\text{free group}\\ \text{on}\\ X_1,\dots,X_n\end{pmatrix} \to \widehat{A}_n = \begin{pmatrix}\text{completed free}\\ \text{associative}\\ \text{algebra on}\\ x_1,\dots,x_n\end{pmatrix}$$

by $X_i \mapsto 1+x_i$ or e^{x_i}

$X_i^{-1} \mapsto 1 - x_i + x_i^2 - \dots$ or e^{-x_i}.

What's "An Expansion"?

A filtration-preserving *isomorphism* $Z: C(G) \to \mathcal{A}(G)$ where

$$I := \{\sum a_i g_i : \sum a_i = 0\} \subset \mathbb{C}G$$

$$\mathbb{C}G = I^0 \supset I^1 \supset I^2 \supset I^3 \supset \cdots$$

$$C(G) := \varprojlim_k \mathbb{C}G/I^k \to \cdots \to \mathbb{C}G/I^2 \to \mathbb{C}G/I \to 0$$

is filtered by $F_m C(G) := \varprojlim_{k>m} I^m/I^k$ and

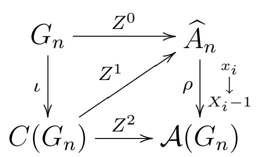
$$\mathcal{A}(G) := \operatorname{gr} C(G) = \hat{\oplus} I^m/I^{m+1}.$$

So all expansions are "equivalent"

Think duals! $C(G)^\star$ are "finite type invariants". $\mathcal{A}(G)^\star$ are "weight systems". Z is a "universal finite type invariant".

$Z_{1,2}$ are Expansions.

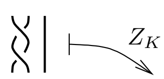
With $Z^0 = Z_1$ or $Z^0 = Z_2$:



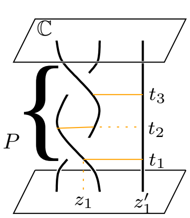
1. ι is automatic.
2. ρ is well-defined.
3. $Z^0|_{I^m} \subset F_m A_n$.
4. Z^0 descends to Z^1.
5. Define Z^2.
6. ρ is surjective.
7. $\operatorname{gr} Z^2$ is the identity.
8. Z^2 is an isomorphism.
9. ρ is an isomorphism.

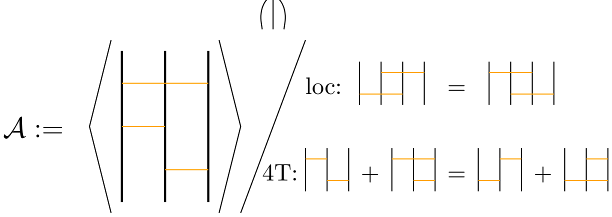
Everything generalizes, step 2 sometimes becomes tricky.

The Kontsevich Integral for Braids



$$\sum_{\substack{m,\ t_1<\dots<t_m\\ P=\{(z_i,z_i')\}}} \frac{D_P}{(2\pi i)^m} \bigwedge_{i=1}^m \frac{dz_i - dz_i'}{z_i - z_i'}$$





M. Kontsevich

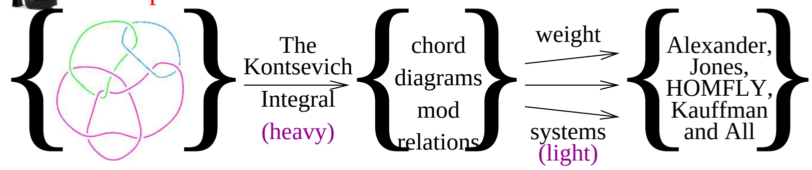
Which other groups / groupoids / categories have expansions?

Dror's Dream / Obsession:

The bigger quest: Understand quantum groups (I don't).

"Unify" quantum groups − find one object that contains all.

Example: One invariant to rule them all:



Easy! Universal! A Morphism! Unique! An Isomorphism!

What is a "Quantum Group"?

For now, a "deformation of the trivial" solution in $\mathcal{U}(\mathfrak{g}))^{\otimes *}[[\hbar]]$ of the major equations:

$$(\Delta\otimes 1)\Delta = (1\otimes\Delta)\Delta \qquad R^{-1}\Delta R = \Delta^{op}$$

$$(\Delta\otimes 1)R = R^{23}R^{13} \qquad (1\otimes\Delta)R = R^{12}R^{13}$$

(as well as a few minor equations).

Why care? Quantum groups computable invariants make!

Visit! katlas.org Edit!

Dror's Guess:

A unified object exists; we'll need:

1. **Expansions as in Lin / universal finite type invariants.**
2. **Naturality / functoriality.**
3. Knotted graphs, especially trivalent.
4. Associators following Drinfel'd.
5. The work of Etingof and Kazhdan on bialgebras.
6. **Virtual braids / knots / knotted graphs.**
7. Polyak (LMP 54) & Haviv (arXiv:math/0211031) on arrow diagrams.

(and when construction ends, we'll dump the scaffolding)

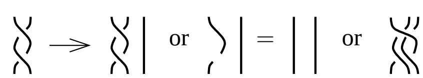
(Quasi?) Natural Expansions

$G \mapsto C(G)$ and $G \mapsto \mathcal{A}(G)$ are functors. Can you choose a ((quasi?) natural) Z satisfying

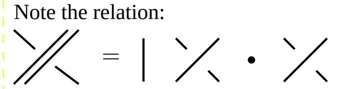
$$\begin{array}{ccc} C(G_1) & \xrightarrow{C(\Delta)} & C(G_2) \\ \downarrow Z(G_1) & G_1 \xrightarrow{\Delta} G_2 & \downarrow Z(G_2) \\ \mathcal{A}(G_1) & \xrightarrow{\mathcal{A}(\Delta)} & \mathcal{A}(G_2) \end{array}$$

(quasi: $\mathcal{A}'(\Delta) = J_\Delta^{-1}\mathcal{A}(\Delta)J_\Delta$? $D_\Delta\mathcal{A}(\Delta)$?)

Perhaps just on a subcategory of **Groups**? Perhaps **Braids** with strands addition, deletion and doubling:



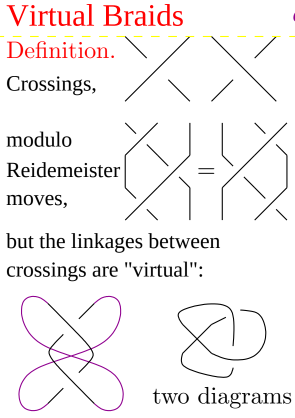
Note the relation:



Virtual Braids

crossings are real, strands go virtual

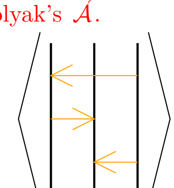
Definition. Crossings, modulo Reidemeister moves, but the linkages between crossings are "virtual":



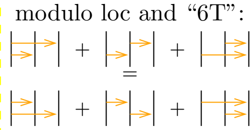
two diagrams of $v2_1$

L. Kauffman

Polyak's $\overrightarrow{\mathcal{A}}$.



modulo loc and "6T":



M. Polyak

T. Ohtsuki

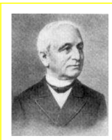
Lie bialgebras. The $\mathfrak{g}$ in a sum $\mathfrak{g}\oplus\mathfrak{g}^\star$ which in itself is a Lie algebra with subalgebras $\mathfrak{g}$ and $\mathfrak{g}^\star$, and in which the tautological metric is invariant.

Why bother? Their deformations are quantum groups, and their diagrammatic universalization is $\overrightarrow{\mathcal{A}}$.

Drinfel'd Etingof Kazhdan Haviv

Question Can you interpret quantum groups as (quasi?)-natural expansions on virtual braids?

Dror's Guess: No, but the effort will be worthwhile.



"God created the knots, all else in topology is the work of mortals"

Leopold Kronecker (modified)