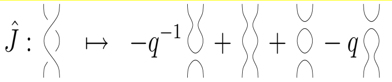
Dror Bar–Natan: Talks: Istanbul–0606

Overview of Khovanov Homology (1)

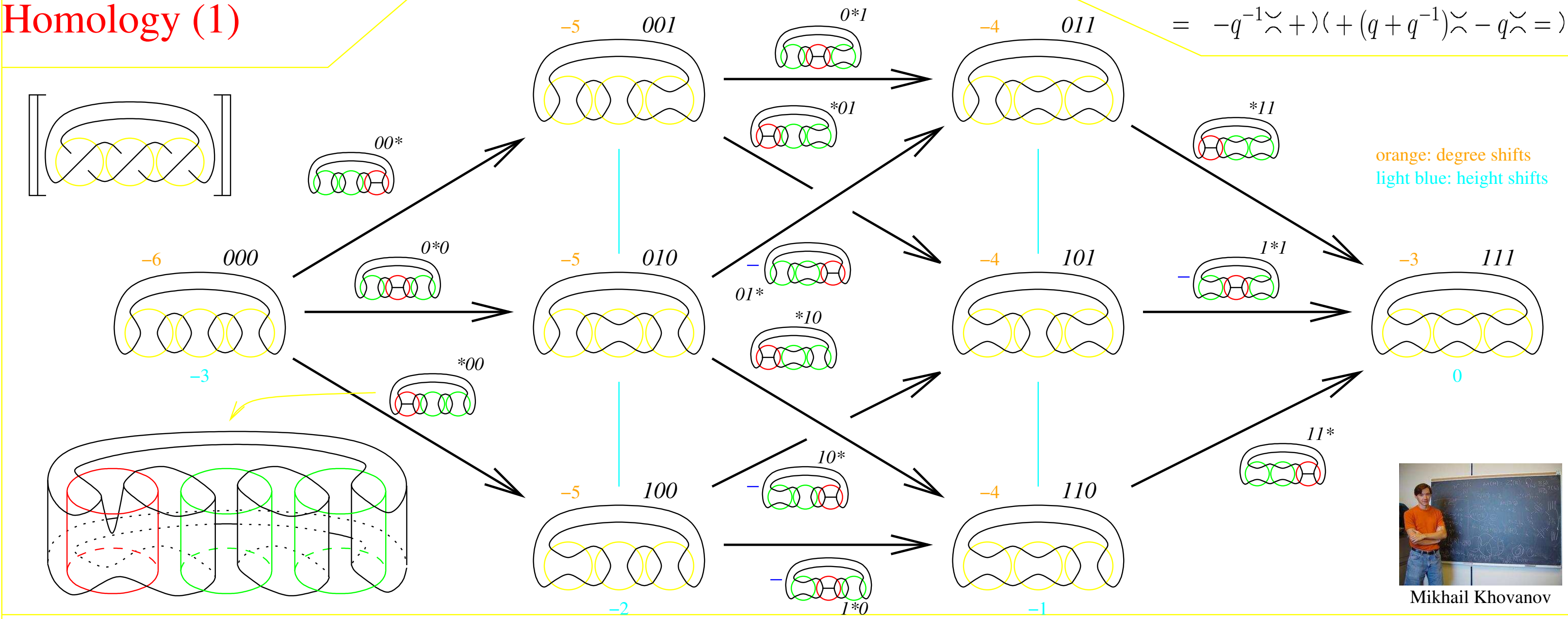
The Jones polynomial: $\bigcirc = q + q^{-1}$

$\hat{J}: \times \mapsto q)(-q^2\asymp,\quad \hat{J}: \times \mapsto -q^{-2}\asymp + q^{-1})($,



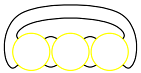
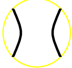
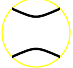
R2

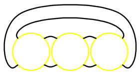
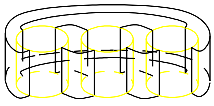
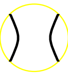
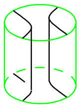
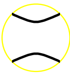
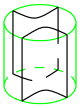
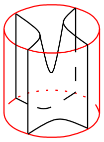
$= -q^{-1}\asymp +)(+ (q+q^{-1})\asymp - q\asymp =)($.



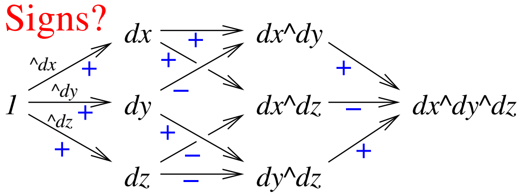
Mikhail Khovanov

What is it?

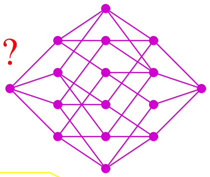
A cube for each knot/link projection; Vertices: All fillings of  with  or with .

Edges: All fillings of $I\times$  =  with $I\times$  =  or with $I\times$  =  and precisely one .

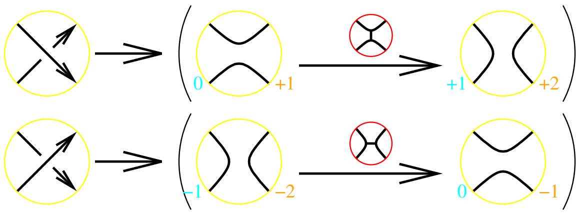
Signs?



More crossings?



General Crossings



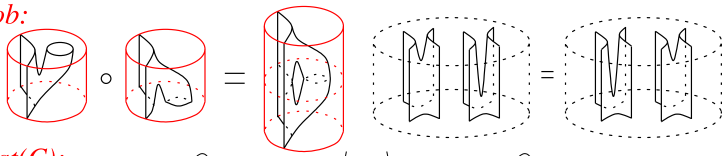
Where does it live?

In *Kom(Mat(<Cob> / {S, T, G, NC})) / homotopy*

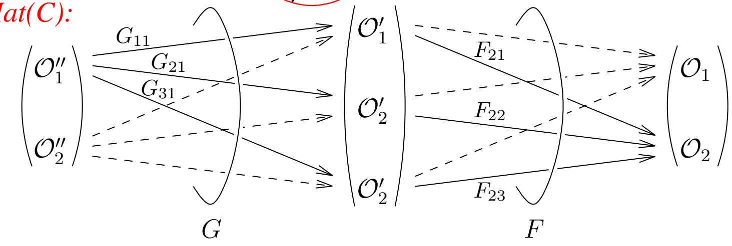
Kom: Complexes Mat: Matrices

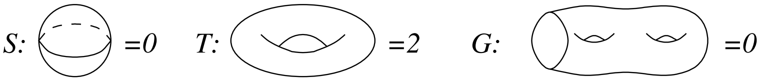
Cob: Cobordisms <...>: Formal lin. comb.

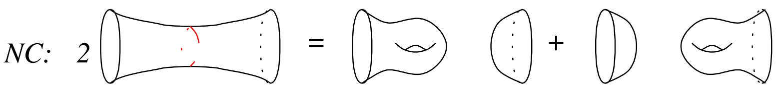
Cob:



Mat(C):

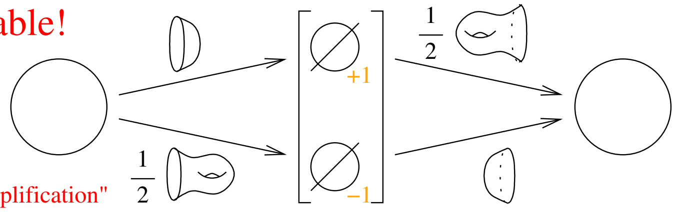






Computable!

via "complex simplification"



Complexes:

$$\Omega = \left(\Omega^{-n_-} \longrightarrow \Omega^{-n_-+1} \longrightarrow \cdots \qquad \longrightarrow \Omega^{n_+}\right)$$

Morphisms:

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega_0^{r-1} \arrow[r, "d^{r-1}"] \arrow[d, "F^{r-1}"'] & \Omega_0^r \arrow[r, "d^r"] \arrow[d, "F^r"'] & \Omega_0^{r+1} \arrow[r] \arrow[d, "F^{r+1}"'] & \cdots \\
\cdots \arrow[r] & \Omega_1^{r-1} \arrow[r, "d^{r-1}"] & \Omega_1^r \arrow[r, "d^r"] & \Omega_1^{r+1} \arrow[r] & \cdots
\end{tikzcd}$$

Homotopies:

$$\begin{tikzcd}
\Omega_0^{r-1} \arrow[r, "d^{r-1}"] \arrow[d, shift right, "F^{r-1}"'] \arrow[d, shift left, "G^{r-1}"] & \Omega_0^r \arrow[r, "d^r"] \arrow[d, shift right, "F^r"'] \arrow[d, shift left, "G^r"] \arrow[dl, "h^r"'] & \Omega_0^{r+1} \arrow[d, shift right, "F^{r+1}"'] \arrow[d, shift left, "G^{r+1}"] \arrow[dl, "h^{r+1}"'] \\
\Omega_1^{r-1} \arrow[r, "d^{r-1}"] & \Omega_1^r \arrow[r, "d^r"] & \Omega_1^{r+1}
\end{tikzcd}$$

$$F^r - G^r = h^{r+1}d^r + d^{r-1}h^r$$

All arrows in an arbitrary additive category!

The Reduction Lemma. If ϕ is an isomorphism then the complex

$$[C] \xrightarrow{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}} \begin{bmatrix} b_1\\ D\end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & \delta\\ \gamma & \epsilon\end{pmatrix}} \begin{bmatrix} b_2\\ E\end{bmatrix} \xrightarrow{\begin{pmatrix}\mu & \nu\end{pmatrix}} [F]$$

is isomorphic to the (direct sum) complex

$$[C] \xrightarrow{\begin{pmatrix}0\\ \beta\end{pmatrix}} \begin{bmatrix} b_1\\ D\end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & 0\\ 0 & \epsilon-\gamma\phi^{-1}\delta\end{pmatrix}} \begin{bmatrix} b_2\\ E\end{bmatrix} \xrightarrow{\begin{pmatrix}0 & \nu\end{pmatrix}} [F]$$