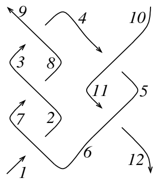
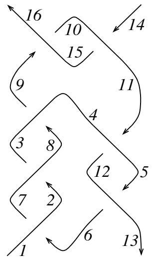
The Conway-Kinoshita-Terasaka Tangles.

Conway Kinoshita Terasaka

$\mathtt{T1} = \mathtt{PD}\left[\bar{X}_{-6,2,7,-1},\ \bar{X}_{-2,8,3,-7},\ \bar{X}_{-8,4,9,-3},\ \bar{X}_{-11,6,12,-5},\ \bar{X}_{-4,11,5,-10}\right];$

$\mathtt{T2} = \mathtt{PD}\left[\bar{X}_{-6,2,7,-1},\ \bar{X}_{-2,8,3,-7},\ \bar{X}_{-8,4,9,-3},\ \bar{X}_{-12,6,13,-5},\ \bar{X}_{-4,12,5,-11},\ \bar{X}_{-10,15,11,-14},\ \bar{X}_{-15,10,16,-9}\right];$





`Column@{TL[T1], Kas[T1]}`

$-2\,\theta\left(u-\frac{\sqrt{3}}{2}\right)+2\,\theta\left(u+\frac{\sqrt{3}}{2}\right)-1$

	$(\gamma_{-10}$	γ_9	γ_{-1}	$\gamma_{12})$
$\bar\gamma_{-10}$	0	$1-\omega$	0	$\omega-1$
$\bar\gamma_{9}$	$\frac{\omega-1}{\omega}$	$\frac{2\omega}{\omega^2-\omega+1}$	$-\frac{\omega-1}{\omega}$	$-\frac{2\omega}{\omega^2-\omega+1}$
$\bar\gamma_{-1}$	0	$\omega-1$	0	$1-\omega$
$\bar\gamma_{12}$	$-\frac{\omega-1}{\omega}$	$-\frac{2\omega}{\omega^2-\omega+1}$	$\frac{\omega-1}{\omega}$	$\frac{2\omega}{\omega^2-\omega+1}$

$-2\,\theta\left(u-\frac{\sqrt{3}}{2}\right)+2\,\theta\left(u+\frac{\sqrt{3}}{2}\right)-1$

	$(\gamma_{-10}$	γ_9	γ_{-1}	$\gamma_{12})$
$\bar\gamma_{-10}$	$2(u-1)(u+1)(4u^2-3)$	0	$-2(u-1)(u+1)(4u^2-3)$	0
$\bar\gamma_{9}$	0	$\frac{1}{2(4u^2-3)}$	0	$-\frac{1}{2(4u^2-3)}$
$\bar\gamma_{-1}$	$-2(u-1)(u+1)(4u^2-3)$	0	$2(u-1)(u+1)(4u^2-3)$	0
$\bar\gamma_{12}$	0	$-\frac{1}{2(4u^2-3)}$	0	$\frac{1}{2(4u^2-3)}$

`Column@{TL[T2], Kas[T2]}`

0

	$(\gamma_{-14}$	γ_{16}	γ_{-1}	$\gamma_{13})$
$\bar\gamma_{-14}$	0	$1-\omega$	0	$\omega-1$
$\bar\gamma_{16}$	$\frac{\omega-1}{\omega}$	$-\frac{2(\omega-1)^2\omega}{\omega^4-3\omega^3+5\omega^2-3\omega+1}$	$-\frac{\omega-1}{\omega}$	$\frac{2(\omega-1)^2\omega}{\omega^4-3\omega^3+5\omega^2-3\omega+1}$
$\bar\gamma_{-1}$	0	$\omega-1$	0	$1-\omega$
$\bar\gamma_{13}$	$-\frac{\omega-1}{\omega}$	$\frac{2(\omega-1)^2\omega}{\omega^4-3\omega^3+5\omega^2-3\omega+1}$	$\frac{\omega-1}{\omega}$	$-\frac{2(\omega-1)^2\omega}{\omega^4-3\omega^3+5\omega^2-3\omega+1}$

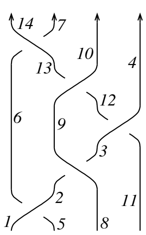
1

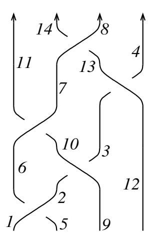
	$(\gamma_{-14}$	γ_{16}	γ_{-1}	$\gamma_{13})$
$\bar\gamma_{-14}$	$\frac{1}{2}(-16u^4+28u^2-13)$	0	$\frac{1}{2}(16u^4-28u^2+13)$	0
$\bar\gamma_{16}$	0	$-\frac{2(u-1)(u+1)}{16u^4-28u^2+13}$	0	$\frac{2(u-1)(u+1)}{16u^4-28u^2+13}$
$\bar\gamma_{-1}$	$\frac{1}{2}(16u^4-28u^2+13)$	0	$\frac{1}{2}(-16u^4+28u^2-13)$	0
$\bar\gamma_{13}$	0	$\frac{2(u-1)(u+1)}{16u^4-28u^2+13}$	0	$-\frac{2(u-1)(u+1)}{16u^4-28u^2+13}$

Examples with non-trivial co-dimension.

$\mathtt{B1} = \mathtt{PD}\left[\bar{X}_{-5,2,6,-1},\ \bar{X}_{-8,3,9,-2},\ X_{-11,4,12,-3},\ X_{-12,10,13,-9},\ \bar{X}_{-13,7,14,-6}\right];$

$\mathtt{B2} = \mathtt{PD}\left[\bar{X}_{-5,2,6,-1},\ \bar{X}_{-9,3,10,-2},\ X_{-10,7,11,-6},\ \bar{X}_{-12,4,13,-3},\ X_{-13,8,14,-7}\right];$





`Column@{TL[B1], Kas[B1]}`

0

	1	0	-1	0	$\frac{1}{\omega}$	0	$-\frac{1}{\omega}$	0
	0	0	0	-1	$\frac{1}{\omega}$	0	$-\frac{1}{\omega}$	1
	$(\gamma_{-11}$	γ_4	γ_{10}	γ_7	γ_{14}	γ_{-1}	γ_{-5}	$\gamma_{-8})$
$\bar\gamma_{-11}$	0	0	0	0	0	0	0	0
$\bar\gamma_{4}$	0	0	0	0	$\frac{\omega-1}{\omega^2}$	0	$-\frac{\omega-1}{\omega^2}$	0
$\bar\gamma_{10}$	0	0	0	0	$-\frac{\omega-1}{\omega}$	0	$\frac{\omega-1}{\omega}$	0
$\bar\gamma_{7}$	0	0	0	0	$\frac{(\omega-1)^2}{\omega^2}$	0	$-\frac{(\omega-1)^2}{\omega^2}$	0
$\bar\gamma_{14}$	0	$-((\omega-1)\omega)$	$\omega-1$	$(\omega-1)^2$	0	$-\frac{\omega-1}{\omega}$	$\frac{\omega-1}{\omega}$	0
$\bar\gamma_{-1}$	0	0	0	0	$\omega-1$	0	$1-\omega$	0
$\bar\gamma_{-5}$	0	$(\omega-1)\omega$	$1-\omega$	$-(\omega-1)^2$	$1-\omega$	$\frac{\omega-1}{\omega}$	$\frac{(\omega-1)^2}{\omega}$	0
$\bar\gamma_{-8}$	0	0	0	0	0	0	0	0

0

	1	0	-1	0	1	0	-1	0
	$(\gamma_{-11}$	γ_4	γ_{10}	γ_7	γ_{14}	γ_{-1}	γ_{-5}	$\gamma_{-8})$
$\bar\gamma_{-11}$	0	0	0	0	0	0	0	0
$\bar\gamma_{4}$	0	0	0	-1	$-u$	0	u	1
$\bar\gamma_{10}$	0	0	0	$-u$	$1-2u^2$	0	$2u^2-1$	u
$\bar\gamma_{7}$	0	-1	$-u$	$2u^2-3$	$-u$	-1	0	1
$\bar\gamma_{14}$	0	$-u$	$1-2u^2$	$-u$	-1	$-u$	$-2(u-1)(u+1)$	u
$\bar\gamma_{-1}$	0	0	0	-1	$-u$	0	u	1
$\bar\gamma_{-5}$	0	u	$2u^2-1$	0	$-2(u-1)(u+1)$	u	$4u^2-3$	0
$\bar\gamma_{-8}$	0	1	u	1	u	1	0	$1-2u^2$

`Column@{TL[B2], Kas[B2]}`

0

	$(\gamma_{-12}$	γ_4	γ_8	γ_{14}	γ_{11}	γ_{-1}	γ_{-5}	$\gamma_{-9})$
$\bar\gamma_{-12}$	$\frac{(\omega-1)^2}{\omega}$	$\omega-1$	$-2(\omega-1)$	$\frac{2(\omega-1)^2}{\omega}$	$\frac{2(\omega-1)}{\omega^2}$	0	$-\frac{2(\omega-1)}{\omega^2}$	$-\frac{(\omega-1)(2\omega-3)}{\omega}$
$\bar\gamma_{4}$	$-\frac{\omega-1}{\omega}$	0	$\frac{\omega-1}{\omega}$	0	0	0	0	0
$\bar\gamma_{8}$	$\frac{2(\omega-1)}{\omega}$	$1-\omega$	$\frac{(\omega-1)^2}{\omega}$	$-\frac{(\omega-1)(2\omega-3)}{\omega}$	$-\frac{2(\omega-1)}{\omega^2}$	0	$\frac{2(\omega-1)}{\omega^2}$	$\frac{2(\omega-2)(\omega-1)}{\omega}$
$\bar\gamma_{14}$	$\frac{2(\omega-1)^2}{\omega}$	0	$-\frac{(\omega-1)(3\omega-2)}{\omega}$	$\frac{3(\omega-1)^2}{\omega}$	$-\frac{(\omega-2)(\omega-1)}{\omega^2}$	0	$-\frac{2(\omega-1)}{\omega^2}$	$-\frac{2(\omega-2)(\omega-1)}{\omega}$
$\bar\gamma_{11}$	$-2(\omega-1)\omega$	0	$2(\omega-1)\omega$	$-((\omega-1)(2\omega-1))$	$\frac{(\omega-1)^2}{\omega}$	$-\frac{\omega-1}{\omega}$	$\frac{2(\omega-1)}{\omega}$	$2(\omega-1)^2$
$\bar\gamma_{-1}$	0	0	0	0	$\omega-1$	0	$1-\omega$	0
$\bar\gamma_{-5}$	$2(\omega-1)\omega$	0	$-2(\omega-1)\omega$	$2(\omega-1)\omega$	$-2(\omega-1)$	$\frac{\omega-1}{\omega}$	$\frac{(\omega-1)^2}{\omega}$	$-((\omega-1)(2\omega-1))$
$\bar\gamma_{-9}$	$-\frac{(\omega-1)(3\omega-2)}{\omega}$	0	$\frac{2(\omega-1)(2\omega-1)}{\omega}$	$-\frac{2(\omega-1)(2\omega-1)}{\omega}$	$\frac{2(\omega-1)^2}{\omega^2}$	0	$-\frac{(\omega-2)(\omega-1)}{\omega^2}$	$\frac{3(\omega-1)^2}{\omega}$

$2\,\theta\left(u-\frac{\sqrt{3}}{2}\right)-2\,\theta\left(u+\frac{\sqrt{3}}{2}\right)$

	1	$\frac{1}{2u}$	0	$-\frac{1}{2u}$	-1	$-\frac{1}{2u}$	0	$\frac{1}{2u}$
	$(\gamma_{-12}$	γ_4	γ_8	γ_{14}	γ_{11}	γ_{-1}	γ_{-5}	$\gamma_{-9})$
$\bar\gamma_{-12}$	0	0	0	0	0	0	0	0
$\bar\gamma_{4}$	0	$-\frac{(2u-1)(2u+1)(2u^2-1)}{4u^2(4u^2-3)}$	$-\frac{2u^2-1}{2u}$	$\frac{1}{4u^2(4u^2-3)}$	0	$-\frac{(2u-1)(2u+1)}{4u^2(4u^2-3)}$	$-\frac{1}{2u(4u^2-3)}$	$\frac{8u^4-6u^2-1}{4u^2(4u^2-3)}$
$\bar\gamma_{8}$	0	$-\frac{2u^2-1}{2u}$	$-2(u-1)(u+1)$	$\frac{2u^2-1}{2u}$	0	$-\frac{1}{2u}$	0	$\frac{1}{2u}$
$\bar\gamma_{14}$	0	$\frac{1}{4u^2(4u^2-3)}$	$\frac{2u^2-1}{2u}$	$\frac{(2u^2-1)(16u^4-16u^2+1)}{4u^2(4u^2-3)}$	0	$-\frac{8u^4-10u^2+1}{4u^2(4u^2-3)}$	$\frac{1}{2u(4u^2-3)}$	$\frac{1}{4u^2(4u^2-3)}$
$\bar\gamma_{11}$	0	0	0	0	0	0	0	0
$\bar\gamma_{-1}$	0	$-\frac{(2u-1)(2u+1)}{4u^2(4u^2-3)}$	$-\frac{1}{2u}$	$-\frac{8u^4-10u^2+1}{4u^2(4u^2-3)}$	0	$\frac{8u^4-10u^2-1}{4u^2(4u^2-3)}$	$\frac{8u^4-10u^2+1}{2u(4u^2-3)}$	$\frac{16u^4-16u^2+1}{4u^2(4u^2-3)}$
$\bar\gamma_{-5}$	0	$-\frac{1}{2u(4u^2-3)}$	0	$\frac{1}{2u(4u^2-3)}$	0	$\frac{8u^4-10u^2+1}{2u(4u^2-3)}$	$\frac{2(u-1)(u+1)(2u-1)(2u+1)}{4u^2-3}$	$\frac{8u^4-6u^2-1}{2u(4u^2-3)}$
$\bar\gamma_{-9}$	0	$\frac{8u^4-6u^2-1}{4u^2(4u^2-3)}$	$\frac{1}{2u}$	$\frac{1}{4u^2(4u^2-3)}$	0	$\frac{16u^4-16u^2+1}{4u^2(4u^2-3)}$	$\frac{8u^4-6u^2-1}{2u(4u^2-3)}$	$-\frac{32u^6-64u^4+30u^2+1}{4u^2(4u^2-3)}$

$$\begin{pmatrix} A & B \\ C & U \end{pmatrix} \xrightarrow{\det(A)} \begin{pmatrix} I & A^{-1}B \\ C & U \end{pmatrix} \xrightarrow{1} \begin{pmatrix} I & A^{-1}B \\ 0 & U-CA^{-1}B \end{pmatrix},$$

so $\det\begin{pmatrix} A & B \\ C & U \end{pmatrix} = \det(A)\det(U-CA^{-1}B)$. (what if $\nexists A^{-1}$?)

Questions. 1. Does this have a topological meaning? 2. Prove the Kashaev conjecture. Is there a version for tangles? 3. Find all solutions of R123 in our "algebra". 4. Braids and the Burau representation. 5. Recover the work in "Prior Art". 6. Are there any concordance properties? 7. What is the "SPQ group"? 8. The jumping points of signatures are the roots of the Alexander polynomial. Does this generalize to tangles? 9. Which of the three Cordon cases is the most common? 10. Are there interesting examples of tangles for which rels is non-trivial? 11. Is the *pq* part determined by Γ-calculus? 12. Is the *pq* part determined by finite type invariants? 13. Does it work with closed components / links? 14. Strand-doubling formulas? 15. A multivariable version? 16. Mutation invariance? 17. Ribbon knots? 18. Are there "face-virtual knots"? 19. Does the pushforward story extend to ranks? To formal Gaussian measures? To super Gaussian measures?

References.

[CC] D. Cimasoni, A. Conway, *Colored Tangles and Signatures,* Math. Proc. Camb. Phil. Soc. **164** (2018) 493–530, arXiv:1507.07818.

[Co] A. Conway, *The Levine-Tristram Signature: A Survey,* arXiv:1903.04477.

[GG] J-M. Gambaudo, É. Ghys, *Braids and Signatures,* Bull. Soc. Math. France **133-4** (2005) 541–579.

[Ka] R. Kashaev, *On Symmetric Matrices Associated with Oriented Link Diagrams,* in *Topology and Geometry, A Collection of Essays Dedicated to Vladimir G. Turaev,* EMS Press 2021, arXiv:1801.04632.

[Me] A. Merz, *An Extension of a Theorem by Cimasoni and Conway,* arXiv:2104.02993.

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