Proof of Theorem 1.

Uniqueness: If A and B are 2 pushforwards, then $\sigma_W(U+A) = \sigma_W(U+B)$ for all PQs U on W.
Thus $\mathcal{D}_A = \mathcal{D}_B$, because otherwise if $w \in \mathcal{D}_A \setminus \mathcal{D}_B$, by taking $U(w) = 1$ on $\mathcal{D}_U = \operatorname{span}\{w\}$, we get $\sigma_W(U+A) = 1 \neq 0 = \sigma_W(U+B)$. Furthermore, A and B must agree where they are both defined, because by taking $U(w) = \frac{-A(w)-B(w)}{2}$ on $\mathcal{D}_U = \operatorname{span}\{w\}$ we get $(U+A)(w) = \frac{A(w)-B(w)}{2} = -(U+B)(w)$, so we must have $A(w) = B(w)$ to satisfy $\sigma_W(U+A) = \sigma_W(U+B)$.

Existence: Define ϕ_*Q by $\mathcal{D}_{\phi_*Q} = \phi(\operatorname{ann}_Q(\ker\phi))$ and $\phi_*Q(w) = Q(v)$ where $v \in \operatorname{ann}_Q(\ker\phi)$. Note that ϕ_*Q is well-defined.
First consider when $U = 0$ on all of W. Let K be a maximal non-degenerate subspace of $\ker\phi$. Then $Q = Q|_K \oplus Q|_{\operatorname{ann}_Q(K)}$, and we can write $\operatorname{ann}_Q(K) = R \oplus A \oplus B$ where $R = \operatorname{rad}_Q(\ker\phi)$ and A, B are chosen so that $A \subseteq \operatorname{ann}_Q(R)$ and $B \subseteq \operatorname{ann}_Q(K) \setminus \operatorname{ann}_Q(R)$. Since $Q : R \to B^*$ is surjective, for any $v \in \mathcal{D}_Q$ there is some $r_v \in R$ such that $Q(r_v, B) = Q(v, B)$. If we choose the r_v so that $r_{v_1} + r_{v_2} = r_{v_1+v_2}$, then we can replace A by $A' = \{a - r_a : a \in A\}$ and B by $B' = \{b - \frac{1}{2}r_b : b \in B\}$ to get $Q = Q|_K \oplus Q|_{R\oplus B'} \oplus Q|_{A'}$.
Then notice that

- $\sigma_V(Q|_K) = \sigma_{\ker\phi}(Q|_{\ker\phi})$
- $\sigma_V(Q|_{R\oplus B'}) = 0$
- $\sigma_V(Q|_{A'}) = \sigma_W(\phi_*Q)$

so we get $\sigma_V(Q) = \sigma_{\ker\phi}(Q|_{\ker\phi}) + \sigma_W(\phi_*Q)$.
Now for an arbitrary U, note that $(Q + \phi^*U)|_{\ker\phi} = Q|_{\ker\phi}$ and $\phi_*(Q + \phi^*U) = \phi_*Q + U$ so we can replace Q in the $U = 0$ case by $Q + \phi^*U$ to get the general case.

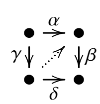
Proof of Theorem 2.

It's clear that pullback is functorial and that pushforward by the identity is the identity. To show $(\phi\psi)_* = \phi_*\psi_*$, use theorem 1 repeatedly to get

$$\begin{aligned}
&\sigma((\phi\psi)_*Q + U)\\
=&\sigma(Q + (\phi\psi)^*U)\\
=&\sigma(Q + \psi^*\phi^*U) - \sigma(Q|_{\ker\phi\psi})\\
=&\sigma(\psi_*Q + \phi^*U) + \sigma(Q|_{\ker\psi}) - \sigma(Q|_{\ker\phi\psi})\\
=&\sigma(\phi_*\psi_*Q + U) + \sigma(Q|_{\ker\psi}) + \sigma(\psi_*Q|_{\ker\phi}) - \sigma(Q|_{\ker\phi\psi})\\
=&\sigma(\phi_*\psi_*Q + U)
\end{aligned}$$

for any U, where the last step uses theorem 1 on $Q|_{\phi\psi}$ with the map $\psi : \ker\phi\psi \to \ker\phi$.

To show $\alpha_*\gamma^* = \beta^*\delta_*$, first note that $\beta^*\beta_*$ is the identity on any PQ since β is injective, so



$$\alpha_*\gamma^*Q = \beta^*(\beta\alpha)_*\gamma^*Q = \beta^*(\delta\gamma)_*\gamma^*Q = \beta^*\delta_*\gamma_*\gamma^*Q$$

As $\beta^*\delta_*\gamma_*\gamma^*Q$ and $\beta^*\delta_*Q$ have the same values where they are both defined, it remains to show that they have the same domain. Since α is surjective and γ is surjective onto $\ker(\delta)$, we see that

$$\beta^{-1}\delta(A) = \beta^{-1}\delta(A \cap \operatorname{im}\gamma)$$

for any subspace A. By taking $A = \operatorname{ann}_Q(\ker\delta)$, the two sides of the equality become the domains of $\beta^*\delta_*Q$ and $\beta^*\delta_*\gamma_*\gamma^*Q$.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/ICERM-2305/