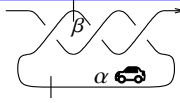


Theorem. The Green function $g_{\alpha\beta}$ is the reading of a traffic counter at β , if car traffic is injected at α (if $\alpha = \beta$, the counter is *after* the injection point).



Example.

$$\sum_{p \geq 0} (1-T)^p = T^{-1} \quad T^{-1} \quad 0 \quad G = \begin{pmatrix} 1 & T^{-1} & 1 \\ 0 & T^{-1} & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Proof. Near a crossing c with sign s , incoming upper edge i and incoming lower edge j , both sides satisfy the g -rules:

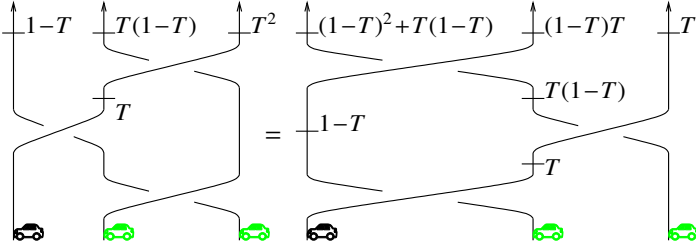
$$g_{i\beta} = \delta_{i\beta} + T^s g_{i+1,\beta} + (1-T^s) g_{j+1,\beta}, \quad g_{j\beta} = \delta_{j\beta} + g_{j+1,\beta},$$

and always, $g_{\alpha,2n+1} = 1$: use common sense and $AG = I (= GA)$.

Bonus. Near c , both sides satisfy the further g -rules:

$$g_{\alpha i} = T^{-s}(g_{\alpha,i+1} - \delta_{\alpha,i+1}), \quad g_{\alpha j} = g_{\alpha,j+1} - (1-T^s)g_{\alpha i} - \delta_{\alpha,j+1}.$$

Invariance of ρ_1 . We start with the hardest, Reidemeister 3:



\Rightarrow Overall traffic patterns are unaffected by Reid3!
 \Rightarrow Green's $g_{\alpha\beta}$ is unchanged by Reid3, provided the cars injection site α and the traffic counters β are away.
 \Rightarrow Only the contribution from the R_1 terms within the Reid3 move matters, and using g -rules the relevant $g_{\alpha\beta}$'s can be pushed outside of the Reid3 area:

$\delta_{i_-,j_-} := \text{If}[i == j, 1, 0]$;
 $gRules_{s_-,i_-,j_-} :=$

$$\{ g_{i\beta_-} \mapsto \delta_{i\beta_-} + T^s g_{i^+,\beta} + (1-T^s) g_{j^+,\beta}, \quad g_{j\beta_-} \mapsto \delta_{j\beta_-} + g_{j^+,\beta}, \\ g_{\alpha,i} \mapsto T^{-s}(g_{\alpha,i^+} - \delta_{\alpha,i^+}), \\ g_{\alpha,j} \mapsto g_{\alpha,j^+} - (1-T^s)g_{\alpha i} - \delta_{\alpha,j^+} \}$$

$lhs = R_1[1, j, k] + R_1[1, i, k^+] + R_1[1, i^+, j^+] // .$

$gRules_{1,j,k} \cup gRules_{1,i,k^+} \cup gRules_{1,i^+,j^+};$

$rhs = R_1[1, i, j] + R_1[1, i^+, k] + R_1[1, j^+, k^+] // .$

$gRules_{1,i,j} \cup gRules_{1,i^+,k} \cup gRules_{1,j^+,k^+};$

Simplify[lhs == rhs]

True

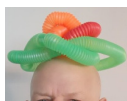
Next comes Reid1, where we use results from an earlier example:

$$R_1[1, 2, 1] - 1 (g_{22} - 1/2) / . \quad g_{\alpha,\beta_-} \mapsto \begin{pmatrix} 1 & T^{-1} & 1 \\ 0 & T^{-1} & 1 \\ 0 & 0 & 1 \end{pmatrix} \llbracket \alpha, \beta \rrbracket$$

$$\frac{1}{T^2} - \frac{1}{T} - \frac{-1 + \frac{1}{T}}{T} = \text{loop}$$

Invariance under the other moves is proven similarly.

Wearing my Topology hat the formula for R_1 , and even the idea to look for R_1 , remain a complete mystery to me.



Wearing my Quantum Algebra hat, I spy a Heisenberg algebra $\mathbb{H} = A\langle p, x \rangle / ([p, x] = 1)$:

cars $\leftrightarrow p$ traffic counters $\leftrightarrow x$

Where did it come from? Consider $g_\epsilon := sl_{2+}^\epsilon := L\langle y, b, a, x \rangle$ with relations

$$[b, x] = \epsilon x, \quad [b, y] = -\epsilon y, \quad [b, a] = 0,$$

$$[a, x] = x, \quad [a, y] = -y, \quad [x, y] = b + \epsilon a.$$

At invertible ϵ , it is isomorphic to sl_2 plus a central factor, and it can be quantized à la Drinfel'd [Dr] much like sl_2 to get an algebra $QU = A\langle y, b, a, x \rangle$ subject to (with $q = e^{\hbar\epsilon}$):

$$[b, a] = 0, \quad [b, x] = \epsilon x, \quad [b, y] = -\epsilon y,$$

$$[a, x] = x, \quad [a, y] = -y, \quad xy - qyx = \frac{1 - e^{-\hbar(b+\epsilon a)}}{\hbar}.$$

Now QU has an R -matrix solving Yang-Baxter (meaning Reid3),

$$R = \sum_{m,n \geq 0} \frac{y^m b^m \otimes (\hbar a)^m (\hbar x)^n}{m! [n]_q!}, \quad ([n]_q! \text{ is a "quantum factorial"})$$

and so it has an associated "universal quantum invariant" à la Lawrence and Ohtsuki [La, Oh], $Z_\epsilon(K) \in QU$.

Now $QU \cong \mathcal{U}(g_\epsilon)$ (only as algebras!) and $\mathcal{U}(g_\epsilon)$ represents into \mathbb{H} via

$$y \rightarrow -tp - \epsilon \cdot xp^2, \quad b \rightarrow t + \epsilon \cdot xp, \quad a \rightarrow xp, \quad x \rightarrow x,$$

(abstractly, g_ϵ acts on its Verma module

$$\mathcal{U}(g_\epsilon) / (\mathcal{U}(g_\epsilon)\langle y, a, b - \epsilon a - t \rangle) \cong \mathbb{Q}[x]$$

by differential operators, namely via \mathbb{H}), so R can be pushed to $\mathcal{R} \in \mathbb{H} \otimes \mathbb{H}$.

Everything still makes sense at $\epsilon = 0$ and can be expanded near $\epsilon = 0$ resulting with $\mathcal{R} = \mathcal{R}_0(1 + \epsilon \mathcal{R}_1 + \dots)$, with $\mathcal{R}_0 = e^{t(xp \otimes 1 - x \otimes p)}$ and \mathcal{R}_1 a quartic polynomial in p and x . So p 's and x 's get created along K and need to be pushed around to a standard location ("normal ordering"). This is done using

$$(p \otimes 1)\mathcal{R}_0 = \mathcal{R}_0(T(p \otimes 1) + (1-T)(1 \otimes p)),$$

$$(1 \otimes p)\mathcal{R}_0 = \mathcal{R}_0(1 \otimes p),$$

and when the dust settles, we get our formulas for ρ_1 . But QU is a quasi-triangular Hopf algebra, and hence ρ_1 is **homomorphic**. Read more at [BV1, BV2] and hear more at $\omega\epsilon\beta/\text{SolvApp}$, $\omega\epsilon\beta/\text{Dogma}$, $\omega\epsilon\beta/\text{DoPeGDO}$, $\omega\epsilon\beta/\text{FDA}$, $\omega\epsilon\beta/\text{AQDW}$.

Also, we can (and know how to) look at higher powers of ϵ and we can (and more or less know how to) replace sl_2 by arbitrary semi-simple Lie algebra (e.g., [Sch]). So ρ_1 is **not alone!**



Schaveling

These constructions are very similar to Rozansky-Overbay [Ro1, Ro2, Ro3, Ov] and hence to the "loop expansion" of the Kontsevich integral and the coloured Jones polynomial.

If this all reads like **insanity** to you, it should (and you haven't seen half of it). Simple things should have simple explanations.

Hence, **Homework**. Explain ρ_1 with no reference to quantum voodoo and find it a topology home (large enough to house generalizations!). Make explicit the homomorphic properties of ρ_1 . Use them to do topology!

P.S. As a friend of Δ , ρ_1 gives a genus bound, sometimes better than Δ 's. How much further does this friendship extend?