

**A Small-Print Page on  $\rho_d, d > 1$ .**

**Definition.**  $\langle f(z_i), h(\zeta_i) \rangle_{\zeta_i=0} := f(\partial_{\zeta_i})h|_{\zeta_i=0}$ , so  $\langle p^2 x^2, \otimes^{8\pi\epsilon} \rangle = 2g^2$ .

**Baby Theorem.** There exist (non unique) power series  $r^\pm(p_1, p_2, x_1, x_2) = \sum_d \epsilon^d r_d^\pm(p_1, p_2, x_1, x_2) \in \mathbb{Q}[T^{\pm 1}, p_1, p_2, x_1, x_2][[\epsilon]]$  with  $\deg r_d^\pm \leq 2d + 2$  ("docile") such that the power series  $Z^b = \sum \rho_d^b \epsilon^d :=$

$$\left\langle \exp\left(\sum_c r^s(p_i, p_j, x_i, x_j)\right), \exp\left(\sum_{\alpha, \beta} g_{\alpha\beta} \pi_\alpha \xi_\beta\right) \right\rangle_{\{p_\alpha, \bar{x}_\beta\}}$$

is a bnot invariant. Beyond the once-and-for-all computation of  $g_{\alpha\beta}$  (a matrix inversion),  $Z^b$  is computable in  $O(n^d)$  operations in the ring  $\mathbb{Q}[T^{\pm 1}]$ .

(Bnots are knot diagrams modulo the braid-like Reidemeister moves, but not the cyclic ones).

**Theorem.** There also exist docile power series  $\gamma^\varphi(\bar{p}, \bar{x}) = \sum_d \epsilon^d \gamma_d^\varphi \in \mathbb{Q}[T^{\pm 1}, \bar{p}, \bar{x}][[\epsilon]]$  such that the power series  $Z = \sum \rho_d \epsilon^d :=$

$$\left\langle \exp\left(\sum_c r^s(p_i, p_j, x_i, x_j) + \sum_k \gamma^{\varphi k}(\bar{p}_k, \bar{x}_k)\right), \exp\left(\sum_{\alpha, \beta} g_{\alpha\beta}(\pi_\alpha + \bar{\pi}_\alpha)(\xi_\beta + \bar{\xi}_\beta) + \sum_\alpha \pi_\alpha \bar{\xi}_\alpha\right) \right\rangle_{\{p_\alpha, \bar{p}_\alpha, \bar{x}_\beta, \bar{\xi}_\beta\}}$$

is a knot invariant, as easily computable as  $Z^b$ .

**Implementation.** Data, then program (with output using the Conway variable  $z = \sqrt{T} - 1 / \sqrt{T}$ ), and then a demo. See `Rho.nb` of `wεβ/ap`.

```
V@r_{1, \varphi}[k_] := \varphi (1/2 - \bar{p}_k \bar{x}_k); V@r_{2, \varphi}[k_] := -\varphi^2 \bar{p}_k \bar{x}_k / 2;
V@r_{3, \varphi}[k_] := -\varphi^3 \bar{p}_k \bar{x}_k / 6
```

```
V@r_{1, s}[i_, j_] :=
s (-1 + 2 p_i x_i - 2 p_j x_j + (-1 + T^5) p_i p_j x_i^2 + (1 - T^5) p_j^2 x_i^2 - 2 p_i p_j x_i x_j + 2 p_j^2 x_i x_j) / 2
```

```
V@r_{2, 1}[i_, j_] :=
(-6 p_i x_i + 6 p_j x_j - 3 (-1 + 3 T) p_i p_j x_i^2 + 3 (-1 + 3 T) p_j^2 x_i^2 + 4 (-1 + T) p_i^2 p_j x_i^3 -
2 (-1 + T) (5 + T) p_i p_j^2 x_i^2 + 2 (-1 + T) (3 + T) p_j^3 x_i^2 + 18 p_i p_j x_i x_j -
18 p_j^2 x_i x_j - 6 p_i^2 p_j x_i^2 x_j + 6 (2 + T) p_i p_j^2 x_i^2 x_j - 6 (1 + T) p_j^3 x_i^2 x_j -
6 p_i p_j^2 x_i x_j^2 + 6 p_j^3 x_i x_j^2) / 12
```

```
V@r_{2, -1}[i_, j_] :=
(-6 T^2 p_i x_i + 6 T^2 p_j x_j + 3 (-3 + T) T p_i p_j x_i^2 - 3 (-3 + T) T p_j^2 x_i^2 -
4 (-1 + T) T p_i^2 p_j x_i^3 + 2 (-1 + T) (1 + 5 T) p_i p_j^2 x_i^2 - 2 (-1 + T) (1 + 3 T) p_j^3 x_i^2 +
18 T^2 p_i p_j x_i x_j - 18 T^2 p_j^2 x_i x_j - 6 T^2 p_i^2 p_j x_i^2 x_j + 6 T (1 + 2 T) p_i p_j^2 x_i^2 x_j -
6 T (1 + T) p_j^3 x_i^2 x_j - 6 T^2 p_i p_j^2 x_i x_j^2 + 6 T^2 p_j^3 x_i x_j^2) / (12 T^2)
```

`Z2[GST48]` (\* takes a few minutes \*)

$$\begin{aligned} & \{1 - 4z^2 - 61z^4 - 207z^6 - 296z^8 - 210z^{10} - 77z^{12} - 14z^{14} - z^{16}, \\ & 1 + (38z^2 + 255z^4 + 1696z^6 + 16281z^8 + 86952z^{10} + 259994z^{12} + 487372z^{14} + 615066z^{16} + 543148z^{18} + 341714z^{20} + \\ & 153722z^{22} + 48983z^{24} + 10776z^{26} + 1554z^{28} + 132z^{30} + 5z^{32}) \epsilon + \\ & (-8 - 484z^2 + 9709z^4 + 165952z^6 + 1590491z^8 + 16256508z^{10} + 115341797z^{12} + 432685748z^{14} + 395838354z^{16} - 4017557792z^{18} - 23300064167z^{20} - \\ & 70082264972z^{22} - 142572271191z^{24} - 209475503700z^{26} - 221616295209z^{28} - 151502648428z^{30} - 23700199243z^{32} + \\ & 99462146328z^{34} + 164920463074z^{36} + 162550825432z^{38} + 119164552296z^{40} + 69153062608z^{42} + 32547596611z^{44} + 12541195448z^{46} + \\ & 3961384155z^{48} + 1021219696z^{50} + 212773106z^{52} + 35264208z^{54} + 4537548z^{56} + 436600z^{58} + 29536z^{60} + 1252z^{62} + 25z^{64}) \epsilon^2\} \end{aligned}$$

`TableForm[Table[Join[{K[[1]][K[[2]]], Z3[K]}, {K, AllKnots[{3, 6]}}], TableAlignments -> Center]` (\* takes a few minutes \*)

$s_1$	$1 + z^2$	$1 + (2z^2 + z^4) \epsilon + (2 - 4z^2 + 3z^4 + 4z^6 + z^8) \epsilon^2 + (-12 + 74z^2 - 27z^4 - 20z^6 + 8z^8 + 6z^{10} + z^{12}) \epsilon^3$
$s_2$	$1 - z^2$	$1 + (-2 + 2z^2) \epsilon^2$
$s_3$	$1 + 3z^2 + z^4$	$1 + (10z^2 + 21z^4 + 12z^6 + 2z^8) \epsilon + (6 - 28z^2 + 33z^4 + 364z^6 + 655z^8 + 536z^{10} + 227z^{12} + 48z^{14} + 4z^{16}) \epsilon^2 + (-60 + 970z^2 + 645z^4 - 3380z^6 - 3280z^8 + 7470z^{10} + 19475z^{12} + 20536z^{14} + 12564z^{16} + 4774z^{18} + 1109z^{20} + 144z^{22} + 8z^{24}) \epsilon^3$
$s_4$	$1 + 2z^2$	$1 + (6z^2 + 5z^4) \epsilon + (4 - 20z^2 + 43z^4 + 64z^6 + 26z^8) \epsilon^2 + (-36 + 498z^2 - 883z^4 + 100z^6 + 816z^8 + 556z^{10} + 146z^{12}) \epsilon^3$
$s_5$	$1 - 2z^2$	$1 + (-2z^2 + z^4) \epsilon + (-4 + 4z^2 + 25z^4 - 8z^6 + 2z^8) \epsilon^2 + (12 - 154z^2 - 223z^4 - 608z^6 - 100z^8 - 52z^{10} + 10z^{12}) \epsilon^3$
$s_6$	$1 - z^2 - z^4$	$1 + (-2z^2 - 3z^4 + 2z^6 + z^8) \epsilon + (-2 - 4z^2 + 29z^4 + 28z^6 + 42z^8 - 8z^{10} - 2z^{12} + 4z^{14} + 2z^{16}) \epsilon^2 + (12 + 166z^2 + 155z^4 - 194z^6 - 2453z^8 - 1622z^{10} - 1967z^{12} - 258z^{14} + 49z^{16} - 30z^{18} + z^{20} + 6z^{22} + z^{24}) \epsilon^3$
$s_7$	$1 + z^2 + z^4$	$1 + (2 + 8z^2 - 16z^4 - 24z^6 - 16z^{10} - 2z^{12}) \epsilon^2$

```
V@r_{3, 1}[i_, j_] :=
(4 p_i x_i - 4 p_j x_j + 2 (5 + 7 T) p_i p_j x_i^2 - 2 (5 + 7 T) p_j^2 x_i^2 - 4 (-5 + 6 T) p_i^2 p_j x_i^3 +
4 (-16 + 17 T + 2 T^2) p_i p_j^2 x_i^2 - 4 (-11 + 11 T + 2 T^2) p_j^3 x_i^2 + 3 (-1 + T) p_i^3 p_j x_i^3 -
3 (-1 + T) (4 + 3 T) p_i^2 p_j^2 x_i^2 + (-1 + T) (13 + 22 T + T^2) p_i p_j^3 x_i^2 -
(-1 + T) (4 + 13 T + T^2) p_j^4 x_i^2 - 28 p_i p_j x_i x_j + 28 p_j^2 x_i x_j + 36 p_i^2 p_j x_i^2 x_j -
12 (9 + 2 T) p_i p_j^2 x_i^2 x_j + 24 (3 + T) p_j^3 x_i^2 x_j - 4 p_i^3 p_j x_i^3 x_j + 28 T p_i^2 p_j^2 x_i^2 x_j -
4 (-6 + 17 T + T^2) p_i p_j^3 x_i^2 x_j + 4 (-5 + 10 T + T^2) p_j^4 x_i^2 x_j + 24 p_i p_j^2 x_i x_j^2 -
24 p_j^3 x_i x_j^2 - 24 p_i^2 p_j^2 x_i^2 x_j^2 + 6 (10 + T) p_i p_j^3 x_i^2 x_j^2 - 6 (6 + T) p_j^4 x_i^2 x_j^2 -
4 p_i p_j^3 x_i x_j^2 + 4 p_j^4 x_i x_j^2) / 24
```

```
V@r_{3, -1}[i_, j_] :=
(-4 T^3 p_i x_i + 4 T^3 p_j x_j - 2 T^2 (7 + 5 T) p_i p_j x_i^2 + 2 T^2 (7 + 5 T) p_j^2 x_i^2 -
4 T^2 (-6 + 5 T) p_i^2 p_j x_i^3 + 4 T (-2 - 17 T + 16 T^2) p_i p_j^2 x_i^2 -
4 T (-2 - 11 T + 11 T^2) p_j^3 x_i^2 + 3 (-1 + T) T^2 p_i p_j x_i^3 - 3 (-1 + T) T (3 + 4 T) p_i^2 p_j^2 x_i^2 +
(-1 + T) (1 + 22 T + 13 T^2) p_i p_j^3 x_i^2 - (-1 + T) (1 + 13 T + 4 T^2) p_j^4 x_i^2 +
28 T^3 p_i p_j x_i x_j - 28 T^3 p_j^2 x_i x_j - 36 T^3 p_i^2 p_j x_i^2 x_j + 12 T^2 (2 + 9 T) p_i p_j^2 x_i^2 x_j -
24 T^2 (1 + 3 T) p_j^3 x_i^2 x_j + 4 T^3 p_i^3 p_j x_i^3 x_j - 28 T^2 p_i^2 p_j^2 x_i^2 x_j -
4 T (-1 - 17 T + 6 T^2) p_i p_j^3 x_i^2 x_j + 4 T (-1 - 10 T + 5 T^2) p_j^4 x_i^2 x_j -
24 T^3 p_i p_j^2 x_i^2 x_j^2 + 24 T^3 p_j^3 x_i x_j^2 + 24 T^2 p_i^2 p_j^2 x_i^2 x_j^2 - 6 T^2 (1 + 10 T) p_i p_j^3 x_i^2 x_j^2 +
6 T^2 (1 + 6 T) p_j^4 x_i^2 x_j^2 + 4 T^3 p_i p_j^3 x_i x_j^2 - 4 T^3 p_j^4 x_i x_j^2) / (24 T^3)
```

```
{p*, x*, p-bar, x-bar} = {pi, epsilon, pi-bar, epsilon-bar}; (z_{i-})^* := (z^*)^i;
Zip_{(i)}[e_] := e;
Zip_{(z, zs...)}[e_] :=
(Collect[e // Zip_{(zs)}, z] /. f_ -> z^{d-} -> (D[f, {z^*, d}])) /. z^* -> 0
```

```
gPair[f_s_, w_] :=
gPair[f_s, w] =
Collect[Zip_{Join@Table[{p_alpha, p_bar_alpha, x_alpha, x_bar_alpha}, {alpha, w}]} [
(Times @@ (V @ f_s))
Exp[Sum[g_{alpha, beta} (pi_alpha + pi_bar_alpha) (epsilon_beta + epsilon_bar_beta), {alpha, w}, {beta, w}] - Sum[epsilon_bar_alpha pi_alpha, {alpha, w}]]],
g_., Factor]
```

```
T2z[p_] := Module[{q = Expand[p], n, c},
If[q == 0, 0, c = Coefficient[q, T, n = Exponent[q, T]];
c z^{2^n} + T2z[q - c (T^{1/2} - T^{-1/2})^{2^n}]]];
```

```
Z_d[K_] := Module[{Cs, phi, n, A, s, i, j, k, delta, G, d1, Z1, Z2, Z3},
{Cs, phi} = Rot[K]; n = Length[Cs]; A = IdentityMatrix[2 n + 1];
Cases[Cs, {s_, i_, j_} -> {A[[{i, j}], {i + 1, j + 1}]] + = (-T^5 T^5 - 1)}];
{delta, G} = Factor@{T^{-Total[phi] - Total[Cs[[All, 1]]]} / 2 Det@A, Inverse@A};
Z1 =
Exp[Total[Cases[Cs, {s_, i_, j_} -> Sum[e^{d1} r_{d1, s}[i, j], {d1, d}]]] +
Sum[e^{d1} gamma_{d1, phi[[k]]}[k], {k, 2 n}], {d1, d}] /. Y_{i, 0}[_] -> 0];
Z2 = Expand[F[{}, {}] x Normal@Series[Z1, {e, 0, d}]] /.
F[f_s_, {e_s_...}] x (f : (r | y)_{ps...}[is_...])^{p-} ->
F[Join[f_s, Table[f, p]], DeleteDuplicates@{e_s, is}];
Z3 = Expand[Z2 /. F[f_s_, e_s_] -> Expand[gPair[
Replace[f_s, Thread[e_s -> Range[Length@e_s], {2}], Length@e_s
] /. g_{alpha, beta} -> G[[e_alpha][e_beta]]]];
Collect[{delta, Z3 /. e^{p-} -> p! delta^{2^p} e^p, epsilon, T2z}];
```