The $sl_3^{\epsilon^2}$ Example (continues Schaveling [Sch]). Here we have two formal variables T_1 and T_2, we set $T_3 := T_1T_2$, we integrate over 6 variables for each edge: p_{1i}, p_{2i}, p_{3i}, x_{1i}, x_{2i}, and x_{3i}.


Schaveling

```
T3 = T1 T2;  i_⁺ := i + 1;
$π =
  (CF@Normal[# + O[ϵ]^2] /.
     {π_is__ :> B^-1 π_is, x_is__ :> B^-1 x_is,
      p_is__ :> B p_is} /. ϵ B^b_ /; b < 0 → 0 /. B → 1) &;

vs_i_ := Sequence[p_{1,i}, p_{2,i}, p_{3,i}, x_{1,i}, x_{2,i}, x_{3,i}];
𝓕[is__] := 𝔼[Sum[π_{ν,i} p_{ν,i}, {i, {is}}, {ν, 3}]];
ℒ[K_] := CF[ℒ /@ Features[K]⟦2⟧];
vs[K_] :=
 Union @@ Table[{vs_i}, {i, Features[K]⟦1⟧}]
```

The Lagrangian.

```
ℒ[X_{i_,j_}[s_]] := T3^s 𝔼[CF@Plus[
   Σ_{ν=1}^3 (x_{νi} (p_{νi⁺} - p_{νi}) + x_{νj} (p_{νj⁺} - p_{νj}) + (T_ν^s - 1) x_{νi} (p_{νi⁺} - p_{νj⁺})),
   (T1^s - 1) p_{3j} x_{1i} (T2^s x_{2i} - x_{2j}),
   ϵ s (T3^s - 1) p_{1j} (p_{2i} - p_{2j}) x_{3i} / (T2^s - 1),
   ϵ s (1/2 + T2^s p_{1i} p_{2j} x_{1i} x_{2i} - p_{1i} p_{2j} x_{1i} x_{2j} - p_{3i} x_{3i} -
      (T2^s - 1) p_{2j} p_{3i} x_{2i} x_{3i} + (T3^s - 1) p_{2j} p_{3j} x_{2i} x_{3i} +
      2 p_{2j} p_{3i} x_{2j} x_{3i} + p_{1i} p_{3j} x_{1i} x_{3j} - p_{2i} p_{3j} x_{2i} x_{3j} -
      T2^s p_{2j} p_{3j} x_{2i} x_{3j} +
      ((T1^s - 1) p_{1j} x_{1i} (T2^{2s} p_{2j} x_{2i} - T2^s p_{2j} x_{2j} -
          (T2^s + 1) (T3^s - 1) p_{3j} x_{3i} + T2^s p_{3j} x_{3j}) +
        (T3^s - 1) p_{3j} x_{3i} (1 - T2^s p_{1i} x_{1i} + p_{2i} x_{2j} + (T2^s - 2) p_{2j} x_{2j})) /
      (T2^s - 1))]]

ℒ[C_{i_}[φ_]] := T3^{φ} 𝔼[Σ_{ν=1}^3 x_{νi} (p_{νi⁺} - p_{νi}) + ϵ φ (p_{3i} x_{3i} - 1/2)]
```

Reidemeister 3.

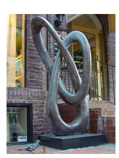
```
Short[
  lhs = ∫ 𝓕[i, j, k] × ℒ /@ (X_{i,j}[1] X_{i⁺,k}[1] X_{j⁺,k⁺}[1])
      𝕕{vs_i, vs_j, vs_k, vs_{i⁺}, vs_{j⁺}, vs_{k⁺}}]
```

```
T1^3 T2^3
 𝔼[3ϵ/2 + T1^2 p_{1,2+i} π_{1,i} - (-1 + T1) T1 p_{1,2+j} π_{1,i} + <<150>>]
```

```
rhs = ∫ 𝓕[i, j, k] × ℒ /@ (X_{j,k}[1] X_{i,k⁺}[1] X_{i⁺,j⁺}[1])
    𝕕{vs_i, vs_j, vs_k, vs_{i⁺}, vs_{j⁺}, vs_{k⁺}};
lhs == rhs
```

```
True
```

The Trefoil.



```
K = Knot[3, 1];  ∫ ℒ[K] 𝕕 vs[K]
```

```
-((i T1^2 T2^2
   𝔼[-((ϵ (1 - T1 + T1^2 - T2 - T1^3 T2 + T2^2 + T1^4 T2^2 - T1 T2^3 -
          T1^4 T2^3 + T1^2 T2^4 - T1^3 T2^4 + T1^4 T2^4)) / ((1 - T1 + T1^2)
         (1 - T2 + T2^2) (1 - T1 T2 + T1^2 T2^2)))]) /
  ((1 - T1 + T1^2) (1 - T2 + T2^2) (1 - T1 T2 + T1^2 T2^2)))
```

A faster program, in which the Feynman diagrams are "pre-computed" (see theta.nb at ωεβ/ap):

```
R1[s_, i_, j_] = CF[
  s (1/2 - g_{3ii} + T2^s g_{1ii} g_{2ji} - g_{1ii} g_{2jj} - (T2^s - 1) g_{2ji} g_{3ii} +
     2 g_{2jj} g_{3ii} - (1 - T3^s) g_{2ji} g_{3ji} - g_{2ii} g_{3jj} - T2^s g_{2ji} g_{3jj} +
     g_{1ii} g_{3jj} +
     ((T1^s - 1) g_{1ji} (T2^{2s} g_{2ji} - T2^s g_{2jj} + T2^s g_{3jj}) +
        (T3^s - 1) g_{3ji} (1 - T2^s g_{1ii} - (T1^s - 1) (T2^s + 1) g_{1ji} +
           (T2^s - 2) g_{2jj} + g_{2ij})) / (T2^s - 1))];

θ[{s0_, i0_, j0_}, {s1_, i1_, j1_}] :=
  CF[s1 (T1^{s0} - 1) (T2^{s1} - 1)^-1 (T3^{s1} - 1) g_{1,j1,i0} g_{3,j0,i1}
    ((T2^{s0} g_{2,i1,i0} - g_{2,i1,j0}) - (T2^{s0} g_{2,j1,i0} - g_{2,j1,j0}))]

Γ1[φ_, k_] = -φ/2 + φ g_{3kk};
```

We call the invariant computed θ:

```
θ[K_] := Module[{Cs, φ, n, A, s, i, j, k, Δ, G, ν, α, β, gEval, c, z},
  {Cs, φ} = Rot[K]; n = Length[Cs];
  A = IdentityMatrix[2 n + 1];
  Cases[Cs, {s_, i_, j_} :>
    (A⟦{i, j}, {i + 1, j + 1}⟧ += ( -T^s  T^s - 1
                                      0     -1   ))];
  Δ = T^((-Total[φ] - Total[Cs⟦All, 1⟧])/2) Det[A];
  G = Inverse[A];
  gEval[ℰ_] := Factor[ℰ /. g_{ν_,α_,β_} :> (G⟦α, β⟧ /. T → T_ν)];
  z = gEval[Σ_{k1=1}^n Σ_{k2=1}^n θ[Cs⟦k1⟧, Cs⟦k2⟧]];
  z += gEval[Σ_{k=1}^n R1 @@ Cs⟦k⟧];
  z += gEval[Σ_{k=1}^{2n} Γ1[φ⟦k⟧, k]];
  {Δ, (Δ /. T → T1) (Δ /. T → T2) (Δ /. T → T3) z} // Factor];
```

Some Knots.

```
Expand[θ[Knot[3, 1]]]
```

```
{-1 + 1/T + T, -1/T1^2 - T1^2 - 1/T2^2 - 1/(T1^2 T2^2) + 1/(T1 T2^2) +
  1/(T1^2 T2) + T1/T2 + T2/T1 + T1^2 T2 - T2^2 + T1 T2^2 - T1^2 T2^2}
```

```
PolyPlot[0] = Graphics[{}];
PolyPlot[p_] := Module[{crs, m1, m2, maxc, minc, s, hex},
  crs = CoefficientRules[T1^(m1=-Exponent[p,T1,Min]) T2^(m2=-Exponent[p,T2,Min]) p,
    {T1, T2}];
  maxc = N@Log@Max@Abs[Last /@ crs];
  minc = N@Log@Min@Select[Abs[Last /@ crs], # > 0 &];
  If[minc == maxc, s[_] = 0,
   s[c_] := s[c] = (maxc - Log@c) / (maxc - minc)];
  hex = Table[{Cos[α], Sin[α]} / Cos[2 π / 12] / 2,
    {α, 2 π / 12, 2 π, 2 π / 6}];
  Graphics[crs /. ({x1_, x2_} → c_) :> {
      If[c == 0, White, Lighter[If[c > 0, Red, Blue],
        0.88 s[Abs@c]]],
      Polygon[(( 1  -1/2
                 0  √3/2 ).{x1 + m1, x2 + m2} + #) & /@ hex] }]];
PolyPlot[{Δ_, θ_}] := PolyPlot[θ]
```

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Beijing-2407 and http://www.math.toronto.edu/~drorbn/Talks/Geneva-2408.