Shifted Partial Quadratics, their Pushforwards, and Signature Invariants for Tangles

Abstract. Following a general discussion of the computation of zombians of unfinished columbaria (with examples), I will tell you about my recent joint work w/ Jessica Liu on what we feel is the "textbook" extension of knot signatures to tangles, which for unknown reasons, is not in any of the textbooks that we know.

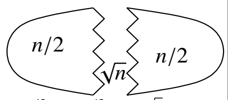

Jessica Liu



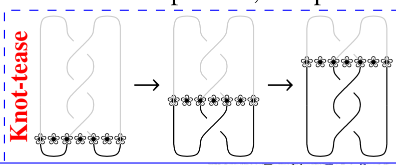
Columbaria in an East Sydney Cemetery — Jacobian, Hamiltonian, Zombian

Prior Art on signatures for tangles / braids. Gambaudo and Ghys [GG], Cimasoni and Conway [CC], Conway [Co], Merz [Me]. All define signatures of tangles / braids by first closing them to links and then work hard to derive composition properties.

Why Tangles? • Faster!
- Conceptually clearer proofs of invariance (and of skein relations).
- Often fun and consequential:
  - ∘ The Jones Polynomial ⇝ The Temperley-Lieb Algebra.
  - ∘ Khovanov Homology ⇝ "Unfinished complexes", complexes in a category.
  - ∘ The Kontsevich Integral ⇝ Associators.
  - ∘ HFK ⇝ OMG, type D, type A, $\mathcal{A}_\infty$, …



$2^{n/2}+2^{n/2}+2^{\sqrt{n}} \ll 2^n$



Zombies: Freepik.com

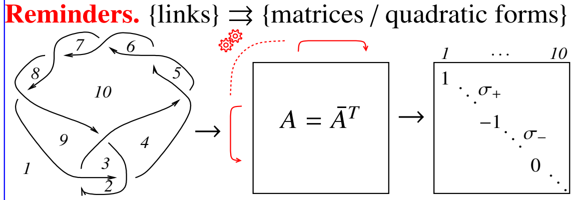
Computing Zombians of Unfinished Columbaria.
- Must be no slower than for finished ones.
- Future zombies must be able to complete the computation.
- Future zombies must not even know the size of the task that today's zombies were facing.
- We must be able to extend to ZPUCs, Zombie Processed Unfinished Columbaria!


Columbarium near Assen

Example / Exercise. Compute the determinant of a $1,000\times 1,000$ matrix in which 50 entries are not yet given.

Homework / Research Projects. • What with ZPUCs? • Use this to get an Alexander tangle invariant.

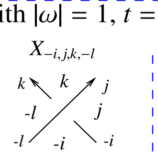
Reminders. {links} ⇉ {matrices / quadratic forms} $\xrightarrow[\sigma]{\text{signature}}$ $\mathbb{Z}$:



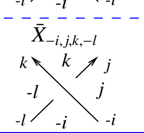
$\to \sigma = \sigma_+ - \sigma_- + s$

With $|\omega|=1$, $t=1-\omega$, $r=t+\bar{t}$, $v=\mathrm{Re}(\omega)$, and $u=\mathrm{Re}(\omega^{1/2})$:

Tristram-Levine (TL) — Kashaev (Kas)



$A \mathrel{+}= \begin{pmatrix} -r & -t & 2t & \bar t \\ -\bar t & 0 & \bar t & 0 \\ 2\bar t & t & -r & -\bar t \\ t & 0 & -t & 0 \end{pmatrix}\begin{matrix} i\\ j\\ k\\ l\end{matrix}$, $s=0$; $\quad A \mathrel{+}= \begin{pmatrix} v & u & 1 & u \\ u & 1 & u & 1 \\ 1 & u & v & u \\ u & 1 & u & 1 \end{pmatrix}\begin{matrix} i\\ j\\ k\\ l\end{matrix}$, $s \mathrel{-}= 1$



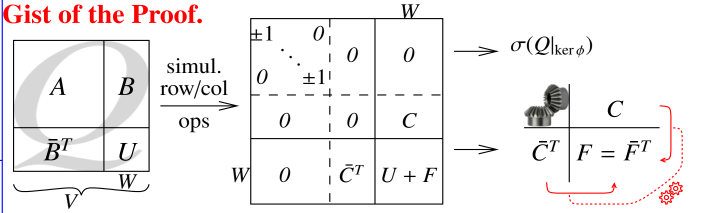
$A \mathrel{+}= \begin{pmatrix} r & -t & -2\bar t & \bar t \\ -\bar t & 0 & \bar t & 0 \\ -2t & t & r & -\bar t \\ t & 0 & -t & 0 \end{pmatrix}\begin{matrix} i\\ j\\ k\\ l\end{matrix}$, $s=0$; $\quad A \mathrel{-}= \begin{pmatrix} v & u & 1 & u \\ u & 1 & u & 1 \\ 1 & u & v & u \\ u & 1 & u & 1 \end{pmatrix}\begin{matrix} i\\ j\\ k\\ l\end{matrix}$, $s \mathrel{+}= 1$

~~**Kashaev's Conjecture**~~ [Ka] For links, $\sigma_{Kas}=2\sigma_{TL}$.
Liu's Theorem [Li].

A *Partial Quadratic (PQ)* on V is a quadratic Q defined only on a subspace $\mathcal{D}_Q\subset V$. We add PQs with $\mathcal{D}_{Q_1+Q_2}:=\mathcal{D}_{Q_1}\cap\mathcal{D}_{Q_2}$. Given a linear $\psi\colon V\to W$ and a PQ Q on W, there is an obvious *pullback* ψ^*Q, a PQ on V.

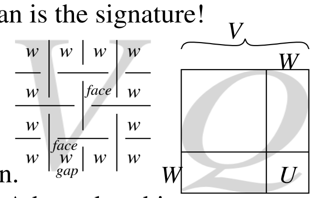
Theorem 1. Given a linear $\phi\colon V\to W$ and a PQ Q on V, there is a unique *pushforward* PQ ϕ_*Q on W such that for every *PQ* U on W, $\sigma_V(Q+\phi^*U)=\sigma_{\ker\phi}(Q|_{\ker\phi})+\sigma_W(U+\phi_*Q)$.
(If you must, $\mathcal{D}(\phi_*Q)=\phi(\mathrm{ann}_Q(\mathcal{D}(Q)\cap\ker\phi))$ and $(\phi_*Q)(w)=Q(v)$, where v is s.t. $\phi(v)=w$ and $Q(v,\mathrm{rad}\,Q|_{\ker\phi})=0$).

Gist of the Proof.



… and the quadratic $F=:\phi_*Q$ is well-defined only on $D:=\ker C$.

Exactly what we want, if the Zombian is the signature!
V: The full space of *faces*.
W: The boundary, made of *gaps*.
Q: The known parts.
U: The part yet unknown.
$\sigma_V(Q+\phi^*(U))$: The overall Zombian.
$\sigma(Q|_{\ker\phi})$: An internal bit. $U+\phi_*Q$: A boundary bit.
And so our ZPUC is the pair $S=(\sigma(Q|_{\ker\phi}),\phi_*Q)$.



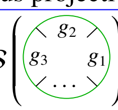
A *Shifted Partial Quadratic (SPQ)* on V is a pair $S=(s\in\mathbb{Z},Q$ a PQ on $V)$. addition also adds the shifts, pullbacks keep the shifts, yet $\phi_*S:=(s+\sigma_{\ker\phi}(Q|_{\ker\phi}),\phi_*Q)$ and $\sigma(S):=s+\sigma(Q)$.

Theorem 1' (*Reciprocity*). Given $\phi\colon V\to W$, for SPQs S on V and U on W we have $\sigma_V(S+\phi^*U)=\sigma_W(U+\phi_*S)$ (and this characterizes ϕ_*S). Note. ψ^* is additive but ϕ_* is not.

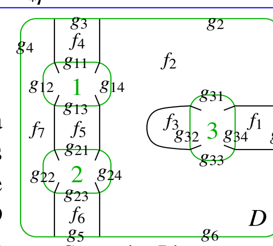
Theorem 2. ψ^* and ϕ_* are functorial.

Theorem 3. "The pullback of a pushforward scene is a pushforward scene": If, on the right, β and δ are arbitrary, $Y=\mathrm{EQ}(\beta,\gamma)=V\oplus_Z W=\{(v,w)\colon \beta v=\gamma w\}$ and μ and ν are the obvious projections, then $\gamma^*\beta_*=\nu_*\mu^*$.

$$\begin{array}{ccc} Y & \xrightarrow{\nu} & W \\ \mu\downarrow & \nearrow & \downarrow\gamma \\ V & \xrightarrow[\beta]{} & Z \end{array}$$

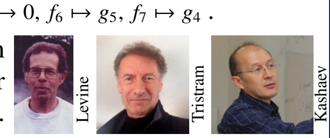
Definition. $\mathcal{S}$  $:=\left\{\begin{matrix}\text{SPQ } S\\ \text{on } \langle g_i\rangle\end{matrix}\right\}$.

Theorem 4. $\{\mathcal{S}(\text{cyclic sets})\}$ is a planar algebra, with compositions $\mathcal{S}(D)((S_i)):=\phi_*^D(\psi_D^*(\bigoplus_i S_i))$, where $\psi_D\colon\langle f_i\rangle\to\langle g_{\alpha i}\rangle$ maps every face of D to the sum of the input gaps adjacent to it and $\phi^D\colon\langle f_i\rangle\to\langle g_i\rangle$ maps every face to the sum of the output gaps adjacent to it. So for our D, $\psi_D\colon$ $f_1\mapsto g_{34}$, $f_2\mapsto g_{31}+g_{14}+g_{24}+g_{33}$, $f_3\mapsto g_{32}$, $f_4\mapsto g_{11}$, $f_5\mapsto g_{13}+g_{21}$, $f_6\mapsto g_{23}$, $f_7\mapsto g_{12}+g_{22}$ and ϕ^D: $f_1\mapsto g_1$, $f_2\mapsto g_2+g_6$, $f_3\mapsto 0$, $f_4\mapsto g_3$, $f_5\mapsto 0$, $f_6\mapsto g_5$, $f_7\mapsto g_4$.



Connection Diagram

Theorem 5. TL and Kas, defined on X and $\bar X$ as before, extend to planar algebra morphisms {tangles} → $\{\mathcal{S}\}$. Restricted to links, $TL=\sigma_{TL}$ and $Kas=\sigma_{Kas}$.



Video: http://www.math.toronto.edu/~drorbn/Talks/Geneva-231201. Handout: http://www.math.toronto.edu/~drorbn/Talks/USC-240205.