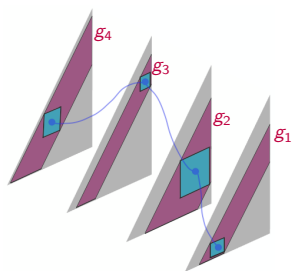


Multi-feathers and multi-sharks.

For a type d invariant we need to count d -tuples of crossings, and each has its own "generation" g_i . So we have the "multi-generation"

$$\vec{g} = (g_1, \dots, g_d).$$

Let $G := \sum g_i$ be the "overall generation". We will choose between a "multi-feather" method and a "multi-shark" method based on the size of G .



Conclusion. We wish to compute the contribution to φ_d coming from d -tuples of crossings of multi-generation \vec{g} .

- ▶ The multi-shark method does it in time

$$\sim (\text{no. of bites}) \cdot (\text{time per bite}) = L^{2d} 2^G \cdot \frac{L}{2^{\min \vec{g}}} < L^{2d+1} 2^G$$

(increases with G).

- ▶ The multi-feather method (project and use the 2D algorithm) does it in time

$$\sim (\text{no. of crossings})^{1/2 d} = \left(\prod_{i=1}^d L^2 \frac{L^2}{2^{g_i}} \right)^{1/2 d} < \frac{L^{3d}}{(2^G)^{3/4}}$$

(decreases with G).

Of course, for any specific G we are free to choose whichever is better, shark or feather.

If time — a word about braids.

Thank You!

The effort to take a single multi-bite is tiny. Indeed,

Lemma Given $2d$ finite sets $B_i = \{t_{i1}, t_{i2}, \dots\} \subset [1..L^3]$ and a permutation $\pi \in S_{2n}$ the quantity

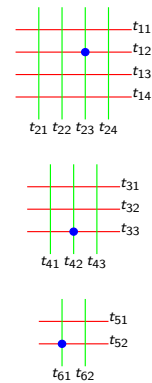
$$N = \left| \left\{ (b_i) \in \prod_{i=1}^{2d} B_i; \text{ the } b_i\text{'s are ordered as } \pi \right\} \right|$$

can be computed in time $\sim \sum |B_i| \sim \max |B_i|$.

Proof. WLOG $\pi = Id$. For $\iota \in [1..2d]$ and $\beta \in B := \cup B_i$ let

$$N_{\iota, \beta} = \left| \left\{ (b_i) \in \prod_{i=1}^{\iota} B_i; b_1 < b_2 < \dots < b_{\iota} \leq \beta \right\} \right|.$$

We need to know $N_{2d, \max B}$; compute it inductively using $N_{\iota, \beta} = N_{\iota, \beta'} + N_{\iota-1, \beta'}$, where β' is the predecessor of β in B . \square



The two methods agree (and therefore are at their worst) if $2^G = L^{\frac{1}{2}(d-1)}$, and in that case, they both take time $\sim L^{\frac{3}{2}d + \frac{3}{2}} = V^{\frac{5}{2}d + \frac{1}{2}}$.

The same reasoning, with the $n^{\frac{2}{3}+e}d$ feather, gives $V^{\frac{4}{3}+e}d$. \square