Some Rigor. (Exercises hints and partial solutions at end)

Exercise 1. Show that if two SPQ's S_1 and S_2 on V satisfy $\sigma(S_1+U) = \sigma(S_2+U)$ for every quadratic U on V, then they have the same shifts and the same domains.

Exercise 2. Show that if two full quadratics Q_1 and Q_2 satisfy $\sigma(Q_1+U) = \sigma(Q_2+U)$ for every U, then $Q_1 = Q_2$.

Proof of Theorem 1'. Fix W and consider triples $(V, S, \phi\colon V \to W)$ where $S = (s, D, Q)$ is an SPQ on V. Say that two triples are "push-equivalent", $(V_1, S_1, \phi_1) \sim (V_2, S_2, \phi_2)$ if for every quadratic U on W,

$$\sigma_{V_1}(S_1 + \phi_1^*U) = \sigma_{V_2}(S_2 + \phi_2^*U).$$

Given our (V, S, ϕ), we need to show:

1. There is an SPQ S' on W such that $(V, S, \phi) \sim (W, S', I)$.
2. If $(W, S', I) \sim (W, S'', I)$ then $S' = S''$.

Property 2 is easy (Exercises 1, 2). Property 1 follows from the following three claims, each of which is easy.

Claim 1. If $v \in \ker\phi \cap D(S)$, and $\lambda := Q(v,v) \neq 0$, then $(V, S, \phi) \sim$

$$\left(V/\langle v\rangle, \left(s+\operatorname{sign}(\lambda), D(S)/\langle v\rangle, Q-\lambda^{-1}Q(-,v)\otimes Q(v,-)\right), \phi/\langle v\rangle\right).$$

So wlog $Q|_{\ker\phi} = 0$ (meaning, $Q|_{\ker\phi\otimes\ker\phi} = 0$). □

Claim 2. If $Q|_{\ker\phi} = 0$ and $v \in \ker\phi \cap D(S)$, let $V' = \ker Q(v,-)$ and then $(V, S, \phi) \sim (V', S|_{V'}, \phi|_{V'})$ so wlog $Q|_{V\otimes\ker\phi+\ker\phi\otimes V} = 0$. □

Claim 3. If $Q|_{V\otimes\ker\phi+\ker\phi\otimes V} = 0$ then $S = \phi^*S'$ for some SPQ S' on $\operatorname{im}\phi$ and then $(V, S, \phi) \sim (W, S', I)$. □ □

Proof of Theorem 2. The functoriality of pullbacks needs no proof. Now assume $V_0 \xrightarrow{\alpha} V_1 \xrightarrow{\beta} V_2$ and that S is an SPQ on V_0. Then for every SPQ U on V_2 we have, using reciprocity three times, that $\sigma(\beta_*\alpha_*S + U) = \sigma(\alpha_*S + \beta^*U) = \sigma(S + \alpha^*\beta^*U) = \sigma(S + (\beta\alpha)^*U) = \sigma((\beta\alpha)_*S + U)$. Hence $\beta_*\alpha_*S = (\beta\alpha)_*S$. □

Definition. A commutative square as on the right is called *admissible* if $\gamma^*\beta_* = \nu_*\mu^*$.

$$\begin{array}{ccc} Y & \xrightarrow{\nu} & W \\ {\scriptstyle\mu}\downarrow & \nearrow & \downarrow{\scriptstyle\gamma} \\ V & \xrightarrow[\beta]{} & Z \end{array}$$

Lemma 1. If $V = W = Y = Z$ and $\beta = \gamma = \mu = \nu = I$, the square is admissible. □

Lemma 2. The following are equivalent:

1. A square as above is admissible.
2. The *Pairing Condition* holds. Namely, if S_1 is an SPQ on V (write $S_1 \vdash V$) and $S_2 \vdash W$, then $\sigma(\mu^*S_1 + \nu^*S_2) = \sigma(\beta_*S_1 + \gamma_*S_2)$.
3. The square is mirror admissible: $\beta^*\gamma_* = \mu_*\nu^*$.

$$\begin{array}{ccccc} & Y & \xrightarrow{\nu} & W & \dashv S_2 \\ & {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\gamma} & \\ S_1 \vdash & V & \xrightarrow[\beta]{} & Z & \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{\nu} & W \\ {\scriptstyle\mu}\downarrow & \swarrow & \downarrow{\scriptstyle\gamma} \\ V & \xrightarrow[\beta]{} & Z \end{array}$$

Proof. Using Exercises 1 and 2 below, and then using reciprocity on both sides, we have $\forall S_1\, \gamma^*\beta_*S_1 = \nu_*\mu^*S_1 \Leftrightarrow \forall S_1\forall S_2\, \sigma(\gamma^*\beta_*S_1+S_2) = \sigma(\nu_*\mu^*S_1+S_2) \Leftrightarrow \forall S_1\forall S_2\, \sigma(\beta_*S_1+\gamma_*S_2) = \sigma(\mu^*S_1 + \nu^*S_2)$, and thus $1 \Leftrightarrow 2$. But the condition in 2 is symmetric under $\beta \leftrightarrow \gamma$, $\mu \leftrightarrow \nu$, so also $2 \Leftrightarrow 3$. □

Lemma 3. If the first diagram below is admissible, then so is the second. □

$$\begin{array}{ccc} Y & \xrightarrow{\nu} & W \\ {\scriptstyle\mu}\downarrow & \nearrow & \downarrow{\scriptstyle\gamma} \\ V & \xrightarrow[\beta]{} & Z \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{\nu} & W \\ {\scriptstyle\mu}\downarrow & \nearrow & \downarrow{\scriptstyle\gamma\oplus 0} \\ V & \xrightarrow[\beta\oplus 0]{} & Z\oplus F \end{array}$$

Lemma 4. A pushforward by an inclusion is the do nothing operation (though note that the pushforward via an inclusion of a fully defined quadratic retains its domain of definition, which now may become partial). □

Lemma 5. For any linear $\phi\colon V \to W$, the diagram on the right is admissible, where ι denotes the inclusion maps.

$$\begin{array}{ccc} V & \xrightarrow{\iota} & V\oplus C \\ {\scriptstyle\phi}\downarrow & \nearrow & \downarrow{\scriptstyle\phi\oplus I} \\ W & \xrightarrow[\iota]{} & W\oplus C \end{array}$$

Proof. Follows easily from Lemma 4. □

Definition. If S is an SPQ with domain D and quadratic Q, the radical of S is the radical of Q considered as a fully-defined quadratic on D. Namely, $\operatorname{rad} S := \{u \in D\colon \forall v \in D,\ Q(u,v) = 0\}$.

Lemma 6. Always, $\phi(\operatorname{rad} S) \subset \operatorname{rad}\phi_*S$.

Proof. Pick $w \in \phi(\operatorname{rad} S)$ and repeat the proof of Theorem 1' but now considering quadruples (V, S, ϕ, v), where (V, S, ϕ) are as before and $v \in \operatorname{rad} S$ satisfies $\phi(v) = w$. Clearly our initial triple (V, S, ϕ) can be extended to such a quadruple, and it is easy to repeat the steps of the proof of Theorem 1' extending everything to such quadruples. □

We have to acknowledge that our proof of Lemma 6 is ugly. We wish we had a cleaner one.

Exercise 3. Show that if two SPQ's S_1 and S_2 on $V \oplus A$ satisfy $A \subset \operatorname{rad} S_i$ and $\sigma(S_1 + \pi^*U) = \sigma(S_2 + \pi^*U)$ for every quadratic U on V, where $\pi\colon V\oplus A \to V$ is the projection, then $S_1 = S_2$.

Exercise 4. Show that if $\phi\colon V \to W$ is surjective and Q is a quadratic on W, then $\sigma(Q) = \sigma(\phi^*Q)$.

Exercise 5. Show that always, $\phi_*\phi^*S = S|_{\operatorname{im}\phi}$.

Lemma 7. For any linear $\phi\colon V \to W$, the diagram on the right is admissible, where $\phi^+ := \phi\oplus I$ and α and β denote the projection maps.

$$\begin{array}{ccc} V\oplus C & \xrightarrow{\phi^+} & W\oplus C \\ {\scriptstyle\alpha}\downarrow & \nearrow & \downarrow{\scriptstyle\beta} \\ V & \xrightarrow[\phi]{} & W \end{array}$$

Proof. Let S be an SPQ on V. Clearly $C \subset \beta^*\phi_*S$. Also, $C \subset \operatorname{rad}\alpha^*S$ so by Lemma 6, $C = \phi^+(C) \subset \phi^+(\operatorname{rad}\alpha^*S) \subset \operatorname{rad}\phi^+_*\alpha^*S$. Hence using Exercise 3, it is enough to show that $\sigma(\phi^+_*\alpha^*S + \beta^*U) = \sigma(\beta^*\phi_*S + \beta^*U)$ for every U on W. Indeed, $\sigma(\phi^+_*\alpha^*S + \beta^*U) \overset{(1)}{=} \sigma(\beta_*\phi^+_*\alpha^*S + U) \overset{(2)}{=} \sigma(\phi_*\alpha_*\alpha^*S + U) \overset{(3)}{=} \sigma(\phi_*S + U) \overset{(4)}{=} \sigma(\beta^*(\phi_*S + U)) \overset{(5)}{=} \sigma(\beta^*\phi_*S + \beta^*U)$, using (1) reciprocity, (2) the commutativity of the diagram and the functoriality of pushing, (3) Exercise 5, (4) Exercise 4, and (5) the additivity of pullbacks. □

Lemma 8. If the first diagram below is admissible, then so are the other two.

$$\begin{array}{ccc} Y & \xrightarrow{\nu} & W \\ {\scriptstyle\mu}\downarrow & \nearrow & \downarrow{\scriptstyle\gamma} \\ V & \xrightarrow[\beta]{} & Z \end{array} \qquad \begin{array}{ccc} Y\oplus E & \xrightarrow{\nu\oplus 0} & W \\ {\scriptstyle\mu\oplus I}\downarrow & \nearrow & \downarrow{\scriptstyle\gamma} \\ V\oplus E & \xrightarrow[\beta\oplus 0]{} & Z \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{\nu\oplus 0} & W\oplus F \\ {\scriptstyle\mu}\downarrow & \nearrow & \downarrow{\scriptstyle\gamma\oplus I} \\ V & \xrightarrow[\beta\oplus 0]{} & Z\oplus F \end{array}$$

Proof. In the diagram

$$\begin{array}{ccccccc} Y\oplus E & \xrightarrow{\pi} & Y & \xrightarrow{\nu} & W & \xrightarrow{\iota} & W\oplus F \\ {\scriptstyle\mu\oplus I}\downarrow & \nearrow & {\scriptstyle\mu}\downarrow & \nearrow & \downarrow{\scriptstyle\gamma} & \nearrow & \downarrow{\scriptstyle\gamma\oplus I} \\ V\oplus E & \xrightarrow[\pi]{} & V & \xrightarrow[\beta]{} & Z & \xrightarrow[\iota]{} & Z\oplus F, \end{array}$$

with π marking projections and ι inclusions, the left square is admissible by Lemma 7, the middle square by assumption, and the right square by Lemma 5. Along with the functoriality of pushforwards this shows the admissibility of both the left and the right 1×2 subrectangles, and these are the diagrams we wanted. □

Proof of Theorem 3. Decompose $Z = A\oplus B\oplus C\oplus D$, where $A = \operatorname{im}\beta\cap\operatorname{im}\gamma$, $\operatorname{im}\beta = A\oplus B$, and $\operatorname{im}\gamma = A\oplus C$. Write $V \simeq A\oplus B\oplus E$ with $\beta = I$ on $A\oplus B$ yet $\beta = 0$ on E, and write $W \simeq A\oplus C\oplus F$ with $\gamma = I$ on $A\oplus C$ yet $\gamma = 0$ on F. Then $Y = V\oplus_Z W \simeq A\oplus E\oplus F$ and our square is as shown on the right, with all maps equal to I on like-named summands and equal to 0 on non-like-named summands. But this diagram is admissible: build it up using Lemma 1 for the A's, and then Lemma 8 for E and C, and then again Lemma 8 along with the mirror property of Lemma 2 for B and F, and then Lemma 3 for D. □

$$\begin{array}{ccc} A\oplus E\oplus F & \longrightarrow & A\oplus C\oplus F \\ \downarrow & \nearrow & \downarrow \\ A\oplus B\oplus E & \longrightarrow & A\oplus B\oplus C\oplus D \end{array}$$

To prove Theorem 4, given three[1] SPQ's S_1, S_2, and S_3, we need to show that planar-multiplying them in two steps, first using a planar connection diagram D_I (I for Inner) to yield $S_6 = \mathcal{S}(D_I)(S_2, S_3)$ and then using a second planar connection diagram D_O (O for Outer) to yield $\mathcal{S}(D_O)(S_1, S_6)$, gives the same answer as multiplying them all at once using the composition planar connection diagram $D_B = D_O \circ_6 D_I$ (B for Big) to yield $\mathcal{S}(D_B)(S_1, S_2, S_3)$.[2] An example should help:

[1] Truly, we need the same for any number of input SPQ's that are divided into two groups, "multiply in the first step" and "multiply in the second step". But there's no added difficulty here, only an added notational complexity.

[2] Aren't we sassy? We picked "6" for the name of the product of "2" and "3".

Video: http://www.math.toronto.edu/~drorbn/Talks/Geneva-231201. Handout: http://www.math.toronto.edu/~drorbn/Talks/USC-240205.