Kashaev's Signature Conjecture

CMS Winter 2021 Meeting, December 4, 2021

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Agenda. Show and tell with signatures.

Abstract. I will display side by side two nearly identical computer programs whose inputs are knots and whose outputs seem to always be the same. I'll then admit, very reluctantly, that I don't know how to prove that these outputs are always the same. One program I wrote mostly in Bedlewo, Poland, in the summer of 2003 and as of recently I understand why it computes the Levine-Tristram signature of a knot. The other is based on the 2018 preprint *On Symmetric Matrices Associated with Oriented Link Diagrams* by Rinat Kashaev (arXiv:1801.04632), where he conjectures that a certain simple algorithm also computes that same signature.

If you can, please turn your video on! (And mic, whenever needed).

http://drorbn.net/cms21

These slides and all the code within are available at http://drorbn.net/cms21.

(I'll post the video there too)

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```
Bed[K_, ω_] :=
 Module[{t, r, XingsByArmpits, bends, faces, p, A, is},
  t = 1 - ω; r = t + t*;
  XingsByArmpits =
   List @@ PD[K] /. x : X[i_, j_, k_, l_] :>
     If[PositiveQ[x], X, X[-i, j, k, -l], X[-j, k, l, -i]];
  bends = Times @@ XingsByArmpits /.
    _[X][a_, b_, c_, d_] :> p_{a,-d} p_{b,-a} p_{c,-b} p_{d,-c};
  faces = bends //. p_{x__,y_} p_{y_,z__} :> p_{x,y,z};
  A = Table[0, Length@faces, Length@faces];
  Do[is = Position[faces, #][[1, 1]] & /@ List @@ x;
   A[is, is] += If[Head[x] === X,
     {{-r, -t, 2t, t*}, {-t*, 0, t*, 0}, {2t*, t, -r, -t*}, {t, 0, -t, 0}},
     -{{r, -t, -2t*, t*}, {-t*, 0, t*, 0}, {-2t, t, r, -t*}, {t, 0, -t, 0}}],
   {x, XingsByArmpits}];
  MatrixSignature[A]];
```

```
Kas[K_, ω_] :=
 Module[{u, v, XingsByArmpits, bends, faces, p, A, is},
  u = Re[ω^(1/2)]; v = Re[ω];
  XingsByArmpits =
   List @@ PD[K] /. x : X[i_, j_, k_, l_] :>
     If[PositiveQ[x], X, X[-i, j, k, -l], X[-j, k, l, -i]];
  bends = Times @@ XingsByArmpits /.
    _[X][a_, b_, c_, d_] :> p_{a,-d} p_{b,-a} p_{c,-b} p_{d,-c};
  faces = bends //. p_{x__,y_} p_{y_,z__} :> p_{x,y,z};
  A = Table[0, Length@faces, Length@faces];
  Do[is = Position[faces, #][[1, 1]] & /@ List @@ x;
   A[is, is] += If[Head[x] === X,
     {{v, u, 1, u}, {u, 1, u, 1}, {1, u, v, u}, {u, 1, u, 1}},
     -{{v, u, 1, u}, {u, 1, u, 1}, {1, u, v, u}, {u, 1, u, 1}}],
   {x, XingsByArmpits}];
  (MatrixSignature[A] - Writhe[K]) / 2];
```

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Why am I showing you 💀code💀?

- I love code — it's fun!
- Believe it or not, it is more expressive than math-talk (though I'll do the math-talk as well, to confirm with prevailing norms).
- It is directly verifiable. Once it is up and running, you'll never ask yourself "did he misplace a sign somewhere"?

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```
Bed[K_, ω_] :=
 Module[{t, r, XingsByArmpits, bends, faces, p, A, is},
  t = 1 - ω; r = t + t*;
  XingsByArmpits =
   List @@ PD[K] /. x : X[i_, j_, k_, l_] :>
     If[PositiveQ[x], X, X[-i, j, k, -l], X[-j, k, l, -i]];
  bends = Times @@ XingsByArmpits /.
    _[X][a_, b_, c_, d_] :> p_{a,-d} p_{b,-a} p_{c,-b} p_{d,-c};
  faces = bends //. p_{x__,y_} p_{y_,z__} :> p_{x,y,z};
  A = Table[0, Length@faces, Length@faces];
  Do[is = Position[faces, #][[1, 1]] & /@ List @@ x;
   A[is, is] += If[Head[x] === X,
     {{-r, -t, 2t, t*}, {-t*, 0, t*, 0}, {2t*, t, -r, -t*}, {t, 0, -t, 0}},
     -{{r, -t, -2t*, t*}, {-t*, 0, t*, 0}, {-2t, t, r, -t*}, {t, 0, -t, 0}}],
   {x, XingsByArmpits}];
  MatrixSignature[A]];
```

```
Kas[K_, ω_] :=
 Module[{u, v, XingsByArmpits, bends, faces, p, A, is},
  u = Re[ω^(1/2)]; v = Re[ω];
  XingsByArmpits =
   List @@ PD[K] /. x : X[i_, j_, k_, l_] :>
     If[PositiveQ[x], X, X[-i, j, k, -l], X[-j, k, l, -i]];
  bends = Times @@ XingsByArmpits /.
    _[X][a_, b_, c_, d_] :> p_{a,-d} p_{b,-a} p_{c,-b} p_{d,-c};
  faces = bends //. p_{x__,y_} p_{y_,z__} :> p_{x,y,z};
  A = Table[0, Length@faces, Length@faces];
  Do[is = Position[faces, #][[1, 1]] & /@ List @@ x;
   A[is, is] += If[Head[x] === X,
     {{v, u, 1, u}, {u, 1, u, 1}, {1, u, v, u}, {u, 1, u, 1}},
     -{{v, u, 1, u}, {u, 1, u, 1}, {1, u, v, u}, {u, 1, u, 1}}],
   {x, XingsByArmpits}];
  (MatrixSignature[A] - Writhe[K]) / 2];
```


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Verification.

```
Once[<< KnotTheory`]
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
MatrixSignature[A_] :=
  Total[Sign[Select[Eigenvalues[A], Abs[#] > 10^-12 &]]];
Writhe[K_] := Sum[If[PositiveQ[x], 1, -1], {x, List @@ PD@K}];
Sum[ω = e^(i RandomReal[{0,2π}]); Bed[K, ω] == Kas[K, ω], {10},
 {K, AllKnots[{3, 10}]}]
```

```
KnotTheory: Loading precomputed data in PD4Knots`.
2490 True
```

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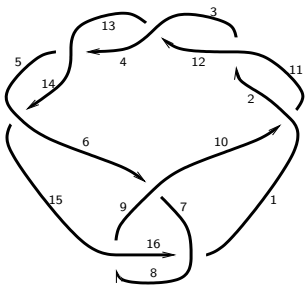
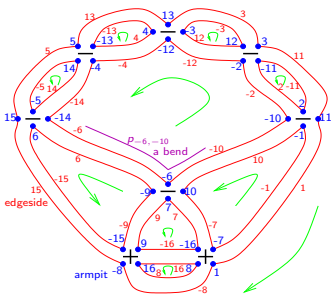
Label everything!



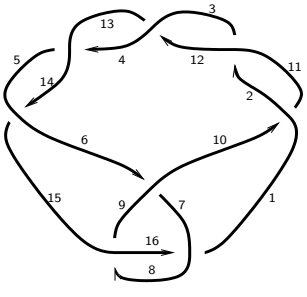
$PD[X[10,1,11,2], X[2,11,3,12], \dots]$



$\{X_-[-1,11,2,-10], X_-[-11,3,12,-2], \dots\}$

Lets run our code line by line...

```
PD[8₂] = PD[X[10, 1, 11, 2],
    X[2, 11, 3, 12], X[12, 3, 13, 4],
    X[4, 13, 5, 14], X[14, 5, 15, 6],
    X[8, 16, 9, 15], X[16, 8, 1, 7],
    X[6, 9, 7, 10]];
K = 8₂;
```



Video and more at http://www.math.toronto.edu/~drorbn/Talks/CMS-2112/