Some Feynman Diagrams in Pure Algebra

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Thanks for allowing me in Columbia U!
ωεβ:=http://drorbn.net/co19/
Slides w/ no handout/URL should be banned!

Abstract. I will explain how the computation of compositions of maps of a certain natural class, from one polynomial ring into another, naturally leads to a certain composition operation of quadratics and to Feynman diagrams. I will also explain, with very little detail, how this is used in the construction of some very well-behaved poly-time computable knot polynomials.

The PBW Principle Lots of algebras are isomorphic as vector spaces to polynomial algebras. So we want to understand arbitrary linear maps between polynomial algebras.

Gentle Agreement. Everything converges!

Convention. For a finite set A, let $z_A := \{z_i\}_{i\in A}$ and let $\zeta_A := \{z_i^* = \zeta_i\}_{i\in A}$. $(y,b,a,x)^* = (\eta,\beta,\alpha,\xi)$

The Generating Series $\mathcal{G}$: $\operatorname{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B]) \to \mathbb{Q}[\![\zeta_A, z_B]\!]$.

Claim. $L\in\operatorname{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B]) \xrightarrow[\mathcal{G}]{\sim} \mathbb{Q}[z_B][\![\zeta_A]\!] \ni \mathcal{L}$ via

$$\mathcal{G}(L) := \sum_{n\in\mathbb{N}^A}\frac{\zeta_A^n}{n!}L(z_A^n) = L\left(\mathbb{e}^{\sum_{a\in A}\zeta_a z_a}\right) = \mathcal{L} = {}_{\text{greek}}\mathcal{L}_{\text{latin}},$$

$$\mathcal{G}^{-1}(\mathcal{L})(p) = \left(p|_{z_a\to\partial_{\zeta_a}}\mathcal{L}\right)_{\zeta_a=0} \quad \text{for } p\in\mathbb{Q}[z_A].$$

Claim. If $L\in\operatorname{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B])$, $M\in\operatorname{Hom}(\mathbb{Q}[z_B]\to\mathbb{Q}[z_C])$, then $\mathcal{G}(L/\!\!/M) = \left(\mathcal{G}(L)|_{z_b\to\partial_{\zeta_b}}\mathcal{G}(M)\right)_{\zeta_b=0}$.

Basic Examples. 1. $\mathcal{G}(id\colon \mathbb{Q}[y,a,x]\to\mathbb{Q}[y,a,x]) = \mathbb{e}^{\eta y+\alpha a+\xi x}$.

2. The standard commutative product m_k^{ij} of polynomials is given by $z_i, z_j \to z_k$. Hence $\mathcal{G}(m_k^{ij}) = m_k^{ij}(\mathbb{e}^{\zeta_i z_i+\zeta_j z_j}) = \mathbb{e}^{(\zeta_i+\zeta_j)z_k}$.

$$\begin{tikzcd} \mathbb{Q}[z]_i\otimes\mathbb{Q}[z]_j \arrow[r, "m_k^{ij}"] \arrow[d, equal] & \mathbb{Q}[z]_k \arrow[d, equal] \\ \mathbb{Q}[z_i,z_j] \arrow[r, "m_k^{ij}"] & \mathbb{Q}[z_k] \end{tikzcd}$$

3. The standard co-commutative co-product Δ^i_{jk} of polynomials is given by $z_i\to z_j+z_k$. Hence $\mathcal{G}(\Delta^i_{jk}) = \Delta^i_{jk}(\mathbb{e}^{\zeta_i z_i}) = \mathbb{e}^{\zeta_i(z_j+z_k)}$.

$$\begin{tikzcd} \mathbb{Q}[z]_i \arrow[r, "\Delta^i_{jk}"] \arrow[d, equal] & \mathbb{Q}[z]_j\otimes\mathbb{Q}[z]_k \arrow[d, equal] \\ \mathbb{Q}[z_i] \arrow[r, "\Delta^i_{jk}"] & \mathbb{Q}[z_j,z_k] \end{tikzcd}$$

Heisenberg Algebras. Let $\mathbb{H} = \langle x,y\rangle/[x,y] = \hbar$ (with $\hbar$ a scalar), let $\mathbb{O}_i\colon \mathbb{Q}[x_i,y_i]\to\mathbb{H}_i$ is the "x before y" PBW ordering map and let hm_k^{ij} be the composition

$$\mathbb{Q}[x_i,y_i,x_j,y_j] \xrightarrow{\mathbb{O}_i\otimes\mathbb{O}_j} \mathbb{H}_i\otimes\mathbb{H}_j \xrightarrow{m_k^{ij}} \mathbb{H}_k \xrightarrow{\mathbb{O}_k^{-1}} \mathbb{Q}[x_k,y_k].$$

Then $\mathcal{G}(hm_k^{ij}) = \mathbb{e}^{\Lambda_\hbar}$, where $\Lambda_\hbar = -\hbar\eta_i\xi_j + (\xi_i+\xi_j)x_k + (\eta_i+\eta_j)y_k$.

Proof 1. Recall the "Weyl form of the CCR" $\mathbb{e}^{\eta y}\mathbb{e}^{\xi x} = \mathbb{e}^{-\hbar\eta\xi}\mathbb{e}^{\xi x}\mathbb{e}^{\eta y}$, and compute

$$\begin{aligned}\mathcal{G}(hm_k^{ij}) &= \mathbb{e}^{\xi_i x_i+\eta_i y_i+\xi_j x_j+\eta_j y_j}/\!\!/\mathbb{O}_i\otimes\mathbb{O}_j/\!\!/m_k^{ij}/\!\!/\mathbb{O}_k^{-1} \\ &= \mathbb{e}^{\xi_i x_i}\mathbb{e}^{\eta_i y_i}\mathbb{e}^{\xi_j x_j}\mathbb{e}^{\eta_j y_j}/\!\!/m_k^{ij}/\!\!/\mathbb{O}_k^{-1} = \mathbb{e}^{\xi_i x_k}\mathbb{e}^{\eta_i y_k}\mathbb{e}^{\xi_j x_k}\mathbb{e}^{\eta_j y_k}/\!\!/\mathbb{O}_k^{-1} \\ &= \mathbb{e}^{-\hbar\eta_i\xi_j}\mathbb{e}^{(\xi_i+\xi_j)x_k}\mathbb{e}^{(\eta_i+\eta_j)y_k}/\!\!/\mathbb{O}_k^{-1} = \mathbb{e}^{\Lambda_\hbar}.\end{aligned}$$

Proof 2. We compute in a faithful 3D representation ρ of $\mathbb{H}$: (ωεβ/hm)

```
{x̂ = (0 1 0
      0 0 0
      0 0 0), ŷ = (0 0 0
                   0 0 ℏ
                   0 0 0), ĉ = (0 0 1
                                0 0 0
                                0 0 0)};
{x̂.ŷ - ŷ.x̂ == ℏ ĉ, x̂.ĉ == ĉ.x̂, ŷ.ĉ == ĉ.ŷ}
{True, True, True}
Λ = -ℏ ηi ξj ck + (ξi + ξj) xk + (ηi + ηj) yk;
Simplify@With[{𝔼 = MatrixExp},
  𝔼[x̂ ξi].𝔼[ŷ ηi].𝔼[x̂ ξj].𝔼[ŷ ηj] ==
   𝔼[x̂ ∂xk Λ].𝔼[ŷ ∂yk Λ].𝔼[ĉ ∂ck Λ]]
True
```

A Real DoPeGDO Example (DoPeGDO:=Docile Perturbed Gaussian Differential Operators). Let $sl^\epsilon_{2+} := L\langle y,b,a,x\rangle$ subject to $[a,x]=x$, $[b,y]=-\epsilon y$, $[a,b]=0$, $[a,y]=-y$, $[b,x]=\epsilon x$, and $[x,y]=\epsilon a+b$. So $t:=\epsilon a-b$ is central and if $\exists\epsilon^{-1}$, $sl^\epsilon_{2+}\cong sl_2\oplus\langle t\rangle$. Let $CU := \mathcal{U}(sl^\epsilon_{2+})$, and let cm_k^{ij} be the composition below, where $\mathbb{O}_i\colon \mathbb{Q}[y_i,b_i,a_i,x_i]\to CU_i$ be the PBW ordering map in the order $ybax$:

$$\begin{tikzcd} CU_i\otimes CU_j \arrow[r, "m_k^{ij}"] & CU_k \\ \mathbb{Q}[y_i,b_i,a_i,x_i,y_j,b_j,a_j,x_j] \arrow[u, "\mathbb{O}_{i,j}"] \arrow[r, "cm_k^{ij}"] & \mathbb{Q}[y_k,b_k,a_k,x_k] \arrow[u, "\mathbb{O}_k"'] \end{tikzcd}$$

Claim. Let (all brawn and no brains)

$$\Lambda = \left(\eta_i + \frac{e^{-\alpha_i-\epsilon\beta_i}\eta_j}{1+\epsilon\eta_j\xi_i}\right)y_k + \left(\beta_i+\beta_j+\frac{\log\left(1+\epsilon\eta_j\xi_i\right)}{\epsilon}\right)b_k +$$
$$\left(\alpha_i+\alpha_j+\log\left(1+\epsilon\eta_j\xi_i\right)\right)a_k + \left(\frac{e^{-\alpha_j-\epsilon\beta_j}\xi_i}{1+\epsilon\eta_j\xi_i}+\xi_j\right)x_k$$

Then $\mathbb{e}^{\eta_i y_i+\beta_i b_i+\alpha_i a_i+\xi_i x_i+\eta_j y_j+\beta_j b_j+\alpha_j a_j+\xi_j x_j}/\!\!/\mathbb{O}_{i,j}/\!\!/cm_k^{ij} = \mathbb{e}^\Lambda/\!\!/\mathbb{O}_k$, and hence $\mathcal{G}(cm_k^{ij}) = \mathbb{e}^\Lambda$.

Proof. We compute in a faithful 2D representation ρ of CU: (ωεβ/sl2)

```
{ŷ = (0 0
      ∈ 0), b̂ = (0  0
                 0 -∈), â = (1 0
                             0 0), x̂ = (0 1
                                        0 0)};
{â.x̂ - x̂.â == x̂, â.ŷ - ŷ.â == -ŷ, b̂.ŷ - ŷ.b̂ == -∈ ŷ,
 b̂.x̂ - x̂.b̂ == ∈ x̂, x̂.ŷ - ŷ.x̂ == b̂ + ∈ â}
{True, True, True, True, True}
Simplify@With[{𝔼 = MatrixExp},
  𝔼[ηi ŷ].𝔼[βi b̂].𝔼[αi â].𝔼[ξi x̂].𝔼[ηj ŷ].𝔼[βj b̂].
   𝔼[αj â].𝔼[ξj x̂] == 𝔼[ŷ ∂yk Λ].𝔼[b̂ ∂bk Λ].𝔼[â ∂ak Λ].
   𝔼[x̂ ∂xk Λ]]
True
Series[Λ, {∈, 0, 2}]
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$$(a_k(\alpha_i+\alpha_j) + y_k(\eta_i + e^{-\alpha_i}\eta_j) + b_k(\beta_i+\beta_j+\eta_j\xi_i) + x_k(e^{-\alpha_j}\xi_i+\xi_j)) +$$
$$\left(a_k\eta_j\xi_i - \frac{1}{2}b_k\eta_j^2\xi_i^2 - e^{-\alpha_i}y_k\eta_j(\beta_i+\eta_j\xi_i) - e^{-\alpha_j}x_k\xi_i(\beta_j+\eta_j\xi_i)\right)\epsilon +$$
$$\left(-\frac{1}{2}a_k\eta_j^2\xi_i^2 + \frac{1}{3}b_k\eta_j^3\xi_i^3 + \frac{1}{2}e^{-\alpha_i}y_k\eta_j\left(\beta_i^2+2\beta_i\eta_j\xi_i+2\eta_j^2\xi_i^2\right) + \frac{1}{2}e^{-\alpha_j}x_k\xi_i\left(\beta_j^2+2\beta_j\eta_j\xi_i+2\eta_j^2\xi_i^2\right)\right)\epsilon^2 + O[\epsilon]^3$$

Note 1. If the lower half of the alphabet (a,b,α,β) is regarded as constants, then $\Lambda = C+Q+\sum_{k\geq 1}\epsilon^k P^{(k)}$ is a docile perturbed Gaussian relative to the upper half of the alphabet (x,y,ξ,η): C is a scalar, Q is a quadratic, and $\deg P^{(k)}\leq 2k+2$.

Note 2. $\operatorname{wt}(x,y,\xi,\eta;a,b,\alpha,\beta;\epsilon) = (1,1,1,1;2,0,0,2;-2)$.

Quadratic Casimirs. If $t\in\mathfrak{g}\otimes\mathfrak{g}$ is the quadratic Casimir of a semi-simple Lie algebra $\mathfrak{g}$, then $\mathbb{e}^t$, regarded by PBW as an element of $\mathcal{S}^{\otimes 2} = \operatorname{Hom}\left(\mathcal{S}(\mathfrak{g})^{\otimes 0}\to\mathcal{S}(\mathfrak{g})^{\otimes 2}\right)$, has a latin-latin dominant Gaussian factor. Likewise for R-matrices.

(Baby) **DoPeGDO** := The category with objects finite sets[†1] and

$$\operatorname{mor}(A\to B) = \{\mathcal{L} = \omega\exp(Q+P)\}\subset\mathbb{Q}[\![\zeta_A,z_B,\epsilon]\!],$$

where: • ω is a scalar.[†2] • Q is a "small" ϵ-free quadratic in $\zeta_A\cup z_B$.[†3] • P is a "docile perturbation": $P=\sum_{k\geq 1}\epsilon^k P^{(k)}$, where $\deg P^{(k)}\leq 2k+2$.[†4] • Compositions:[†6] $\mathcal{L}/\!\!/\mathcal{M} := \left(\mathcal{L}|_{z_i\to\partial_{\zeta_i}}\mathcal{M}\right)_{\zeta_i=0}$.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Columbia-191125/