I Still Don't Understand the Alexander Polynomial

Dror Bar-Natan, http://drorbn.net/mo21

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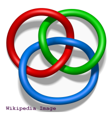
Abstract. As an algebraic knot theorist, I still don't understand the Alexander polynomial. There are two conventions as for how to present tangle theory in algebra: one may name the strands of a tangle, or one may name their ends. The distinction might seem too minor to matter, yet it leads to a completely different view of the set of tangles as an algebraic structure. There are lovely formulas for the Alexander polynomial as viewed from either perspective, and they even agree where they meet. But the "strands" formulas know about strand doubling while the "ends" ones don't, and the "ends" formulas know about skein relations while the "strands" ones don't. There ought to be a common generalization, but I don't know what it is.

I use talks to self-motivate; so often I choose a topic and write an abstract when I know I can do it, yet when I haven't done it yet. This time it turns out my abstract was wrong — I'm still uncomfortable with the Alexander polynomial, but in slightly different ways than advertised two slides before.

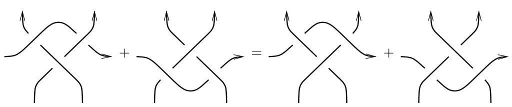
My discomfort.

- I can compute the multivariable Alexander polynomial real fast:



$\longrightarrow (uvw)^{-1/2}(u-1)(v-1)(w-1).$

- But I can only prove "skein relations" real slow:



1. Virtual Skein Theory Heaven

Definition. A "Contraction Algebra" assigns a set $\mathcal{T}(\mathcal{X}, X)$ to any pair of finite sets $\mathcal{X} = \{\xi \dots\}$ and $X = \{x, \dots\}$ provided $|\mathcal{X}| = |X|$, and has operations

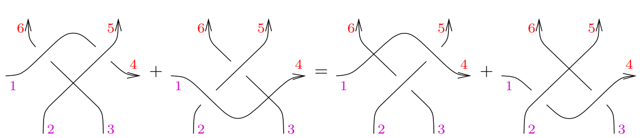
- "Disjoint union" $\sqcup\colon \mathcal{T}(\mathcal{X}, X) \times \mathcal{T}(\mathcal{Y}, Y) \to \mathcal{T}(\mathcal{X} \sqcup \mathcal{Y}, X \sqcup Y)$, provided $\mathcal{X} \cap \mathcal{Y} = X \cap Y = \emptyset$.
- "Contractions" $c_{x,\xi}\colon \mathcal{T}(\mathcal{X}, X) \to \mathcal{T}(\mathcal{X} \setminus \xi, X \setminus x)$, provided $x \in X$ and $\xi \in \mathcal{X}$.
- Renaming operations $\sigma^\xi_\eta\colon \mathcal{T}(\mathcal{X} \sqcup \{\xi\}, X) \to \mathcal{T}(\mathcal{X} \sqcup \{\eta\}, X)$ and $\sigma^x_y\colon \mathcal{T}(\mathcal{X}, X \sqcup \{x\}) \to \mathcal{T}(\mathcal{X}, X \sqcup \{y\})$.

Subject to axioms that will be specified right after the two examples in the next three slides.

If R is a ring, a contraction algebra is said to be "R-linear" if all the $\mathcal{T}(\mathcal{X}, X)$'s are R-modules, if the disjoint union operations are R-bilinear, and if the contractions $c_{x,\xi}$ and the renamings $\sigma^\cdot_\cdot$ are R-linear.

(Contraction algebras with some further "unit" properties are called "wheeled props" in [MMS, DHR])

Note 3. A contraction algebra morphism out of $\mathcal{T}$ is an invariant of virtual tangles (and hence of virtual knots and links) and would be an ideal tool to prove Skein Relations:



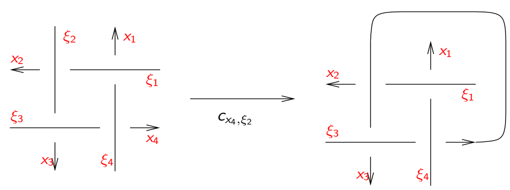
Thanks for inviting me to Moscow! As most of you have never seen it, here's a picture of the lecture room:

If you can, please turn your video on! (And mic, whenever needed).

This talk is to a large extent an elucidation of the Ph.D. theses of my former students Jana Archibald and Iva Halacheva. See [Ar, Ha1, Ha2].

Also thanks to Roland van der Veen for comments.

A technicality. There's supposed to be fire alarm testing in my building today. Don't panic!



Example 1. Let $\mathcal{T}(\mathcal{X}, X)$ be the set of virtual tangles with incoming ends ("tails") labeled by $\mathcal{X}$ and outgoing ends ("heads") labeled by X, with $\sqcup$ and $\sigma^\cdot_\cdot$ the obvious disjoint union and end-renaming operations, and with $c_{x,\xi}$ the operation of attaching a head x to a tail ξ while introducing no new crossings.

Note 1. $\mathcal{T}$ can be made linear by allowing formal linear combinations.

Note 2. $\mathcal{T}$ is finitely presented, with generators the positive and negative crossings, and with relations the Reidemeister moves! (If you want, you can take this to be the definition of "virtual tangles").

Example 2. Let V be a finite dimensional vector space and set $\mathcal{V}(\mathcal{X}, X) := (V^*)^{\otimes \mathcal{X}} \otimes V^{\otimes X}$, with $\sqcup = \otimes$, with $\sigma^\cdot_\cdot$ the operation of renaming a factor, and with $c_{x,\xi}$ the operation of contraction: the evaluation of tensor factor ξ (which is a V^*) on tensor factor x (which is a V).

Video and more at http://www.math.toronto.edu/~drorbn/Talks/MoscowByWeb-2104/