



Strand Doubling and Reversal.

$$\omega \begin{array}{c|cc} a & S & \\ \hline S & \alpha & \theta \\ \hline a & \phi & \Xi \end{array} \xrightarrow{\substack{\mu = T_a^{-1} \\ \mu = T_a^{-1} \\ T_a = T_a^{-1}}} \left(\begin{array}{c|cc} \omega & b & c \\ \hline b & (\sigma_a - \alpha T_a - \nu T_c) / \mu & (T_b - 1) T_c \nu / \mu \\ \hline c & (T_c - 1) \nu / \mu & (\alpha - \sigma_a T_a - \nu T_c) / \mu \\ \hline S & \phi & \Xi \end{array} \right)$$

Where σ assigns to every $a \in S$ a Laurent monomial σ_a in $\{t_b\}_{b \in S}$ subject to $\sigma(a \nearrow_b, b \nearrow_a) = (a \rightarrow 1, b \rightarrow t_a^{\pm 1})$, $\sigma(T_1 \sqcup T_2) = \sigma(T_1) \sqcup \sigma(T_2)$, and $\sigma // m_c^{ab} = (\sigma \setminus \{a, b\}) \cup (c \rightarrow \sigma_a \sigma_b)_{t_a, t_b \rightarrow t_c}$.

Vo's Thesis [Vo]. A proof of the Fox-Milnor theorem for ribbon knots using this technology (and more).

Implementation key idea: ωεβ/AlexDemo

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(ω, A = (αab)) ↔
(ω, λ = ∑ αab ta hb)

Γ := F[ω, λ] / F[ω, λ] := F[ω, λ] / F[ω, λ]
ma,b := Γ[ω, λ] := Module[{a, b, δ, θ, e, φ, ψ, Ξ, μ},
  {α β θ} = {
    θa,b ha,λ θc,b hb,λ θc,λ
    θb,b hb,λ θc,b hb,λ θc,λ
    θb,b hb,λ θc,b hb,λ λ
  } /. {t} ha,b → 0;
  Γ[(μ - 1 - β) ω, (ta, 1) · {γ + α δ / μ + e + δ θ / μ} · {ha, 1}]
  /. {Ta → Ta, Tb → Tc} // FCollect];
M = OuterFactor[θa,b ha,λ] & {S, S};
M = Prepend[M, ta & / θ &] // Transpose;
M = Prepend[M, Prepend[ha & / θ &, ω]];
M // MatrixForm;
Rma,b := Rpa,b /. Ta → 1 / Ta;

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Meta-Associativity

$$\xi = \Gamma[\omega, \{t_1, t_2, t_3, t_S\}] \cdot \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \theta_1 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \theta_2 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \theta_3 \\ \phi_1 & \phi_2 & \phi_3 & \Xi \end{pmatrix} \cdot \{h_1, h_2, h_3, h_S\};$$

$$(\xi // m_{1 \rightarrow 2} // m_{1 \rightarrow 3}) = (\xi // m_{2 \rightarrow 3} // m_{1 \rightarrow 2})$$

True R3 ... divide and conquer!

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{Rm51 Rm62 Rp34 // m14→1 // m25→2 // m36→3,
 Rp61 Rm24 Rm35 // m14→1 // m25→2 // m36→3}

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$$\begin{pmatrix} 1 & h_1 & h_2 & h_3 \\ t_1 & \frac{T_3}{T_2} & 0 & 0 \\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0 \\ t_3 & \frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1 \end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 & h_3 \\ t_1 & \frac{T_3}{T_2} & 0 & 0 \\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0 \\ t_3 & \frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1 \end{pmatrix}$$

$$z = Rm_{12,1} Rm_{27} Rm_{63} Rm_{4,11} Rp_{16,5} Rp_{6,13} Rp_{14,9} Rp_{10,15};$$

$$D0[z = z // m_{1 \rightarrow 1}, \{k, 2, 16\}];$$

$$z = \begin{pmatrix} 11 - \frac{1}{T_1^2} + \frac{4}{T_1} - \frac{8}{T_1} - 8 T_1 + 4 T_1^2 - T_1^3 & h_1 \\ & 1 \end{pmatrix}$$

Fact. Γ is better viewed as an invariant of a certain class of 2D knotted objects in \mathbb{R}^4 [BND, BN].

Fact. Γ is the “0-loop” part of an invariant that generalizes to “n-loops” (1D tangles only, see further talks and future publications with van der Veen).

Speculation. Stepping stones to categorification?

Ask me about geography vs. identity!

[BN] D. Bar-Natan, *Balloons and Hoops and their Universal* **References.**

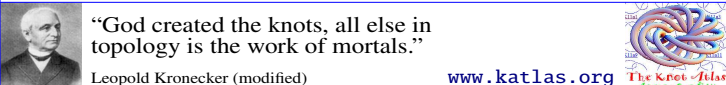
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[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I: w-Knots and the Alexander Polynomial*, Alg. and Geom. Top. **16-2** (2016) 1063–1133, arXiv:1405.1956, ωεβ/WK01.

[BNS] D. Bar-Natan and S. Selmani, *Meta-Monoids, Meta-Bicrossed Products, and the Alexander Polynomial*, J. of Knot Theory and its Ramifications **22-10** (2013), arXiv:1302.5689.

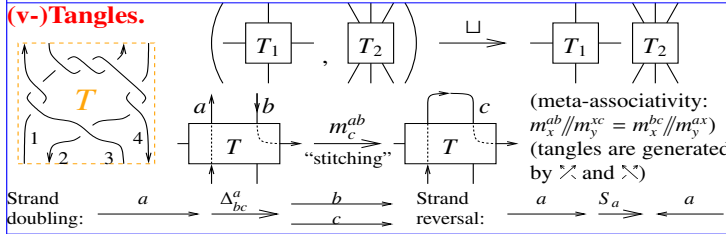
[GST] R. E. Gompf, M. Scharlemann, and A. Thompson, *Fibered Knots and Potential Counterexamples to the Property 2R and Slice-Ribbon Conjectures*, Geom. and Top. **14** (2010) 2305–2347, arXiv:1103.1601.

[Vo] H. Vo, *Alexander Invariants of Tangles via Expansions*, University of Toronto Ph.D. thesis, ωεβ/Vo.



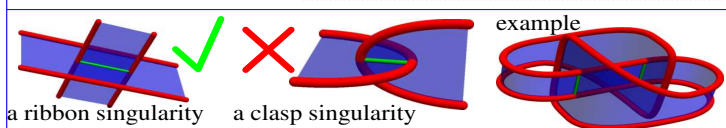
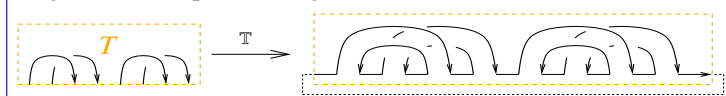
Algebraic Knot Theory

Abstract. This will be a very “light” talk: I will explain why about 13 years ago, in order to have a say on some problems in knot theory, I’ve set out to find tangle invariants with some nice compositional properties. In other talks in Sydney (ωεβ/talks) I have explained / will explain how such invariants were found - though they are yet to be explored and utilized.



Genus. Every knot is the boundary of an orientable “Seifert Surface” (ωεβ/SS), and the least of their genera is the “genus” of the knot.

Claim. The knots of genus ≤ 2 are precisely the images of 4-component tangles via

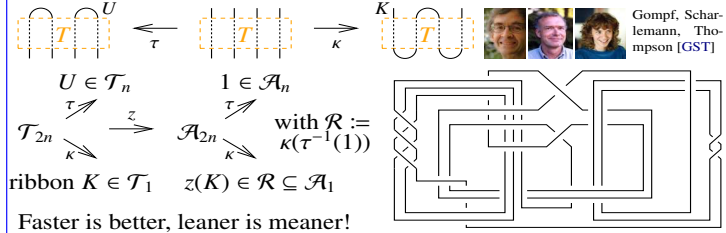


A Bit about Ribbon Knots. A “ribbon knot” is a knot that can be presented as the boundary of a disk that has “ribbon singularities”, but no “clasp singularities”. A “slice knot” is a knot in $S^3 = \partial B^4$ which is the boundary of a non-singular disk in B^4 . Every ribbon knot is clearly slice, yet,

Conjecture. Some slice knots are not ribbon.

Fox-Milnor. The Alexander polynomial of a ribbon knot is always of the form $A(t) = f(t)f(1/t)$. (also for slice)

Theorem. K is ribbon iff it is κT for a tangle T for which τT is the untangle U .



The Gold Standard is set by the “ Γ -calculus” Alexander formulas [BNS, BN]. An S -component tangle T has

$$\Gamma(T) \in R_S \times M_{S \times S}(R_S) = \left\{ \begin{array}{c|c} \omega & S \\ \hline S & A \end{array} \right\} \text{ with } R_S := \mathbb{Z}\langle \{T_a : a \in S\} \rangle;$$

$$(a \nearrow_b, b \nearrow_a) \rightarrow \begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & -T_a^{-1} \\ b & 0 & T_a^{-1} \end{array} \quad T_1 \sqcup T_2 \rightarrow \begin{array}{c|cc} \omega_1 \omega_2 & S_1 & S_2 \\ \hline S_1 & A_1 & 0 \\ S_2 & 0 & A_2 \end{array}$$

$$\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow{m_c^{ab}} \begin{array}{c|cc} (1-\beta)\omega & c & S \\ \hline c & \gamma + \frac{\alpha\delta}{1-\beta} & \epsilon + \frac{\delta\theta}{1-\beta} \\ S & \phi + \frac{\alpha\psi}{1-\beta} & \Xi + \frac{\psi\theta}{1-\beta} \end{array}$$

For long knots, ω is Alexander, and that’s the fastest Alexander algorithm I know! Dunfield: 1000-crossing fast.