

Axioms. One axiom is primary and interesting,

- ▶ Contractions commute! Namely, $c_{x,\xi} \parallel c_{y,\eta} = c_{y,\eta} \parallel c_{x,\xi}$ (or in old-speak, $c_{y,\eta} \circ c_{x,\xi} = c_{x,\xi} \circ c_{y,\eta}$).

And the rest are just what you'd expect:

- ▶ \sqcup is commutative and associative, and it commutes with $c_{\cdot,\cdot}$ and with σ_{\cdot} whenever that makes sense.
- ▶ $c_{\cdot,\cdot}$ is "natural" relative to renaming: $c_{x,\xi} = \sigma_y^x \parallel \sigma_\eta^\xi \parallel c_{y,\eta}$.
- ▶ $\sigma_\xi^\xi = \sigma_x^x = Id$, $\sigma_\eta^\xi \parallel \sigma_\zeta^\eta = \sigma_\zeta^\xi$, $\sigma_y^x \parallel \sigma_z^y = \sigma_z^x$, and renaming operations commute where it makes sense.

2. Heaven is a Place on Earth

(A version of the main results of Archibald's thesis, [Ar]).

Let us work over the base ring $\mathcal{R} = \mathbb{Q}[\{T^{\pm 1/2} : T \in C\}]$. Set

$$\mathcal{A}(\mathcal{X}, X) := \{w \in \Lambda(\mathcal{X} \sqcup X) : \deg_{\mathcal{X}} w = \deg_X w\}$$

(so in particular the elements of $\mathcal{A}(\mathcal{X}, X)$ are all of even degree). The union operation is the wedge product, the renaming operations are changes of variables, and $c_{x,\xi}$ is defined as follows. Write $w \in \mathcal{A}(\mathcal{X}, X)$ as a sum of terms of the form uw' where $u \in \Lambda(\xi, x)$ and $w' \in \mathcal{A}(\mathcal{X} \setminus \xi, X \setminus x)$, and map u to 1 if it is 1 or $x\xi$ and to 0 if it is ξ or x :

$$1w' \mapsto w', \quad \xi w' \mapsto 0, \quad xw' \mapsto 0, \quad x\xi w' \mapsto w'.$$

Proposition. \mathcal{A} is a contraction algebra.

We construct a morphism of coloured contraction algebras $\mathcal{A} : \mathcal{T} \rightarrow \mathcal{A}$ by declaring

$$\begin{aligned} X_{ijkl}[S, T] &\mapsto T^{-1/2} \exp \left((\xi_i \ \xi_j) \begin{pmatrix} 1 & 1-T \\ 0 & T \end{pmatrix} \begin{pmatrix} x_j \\ x_k \end{pmatrix} \right) \\ \bar{X}_{ijkl}[S, T] &\mapsto T^{1/2} \exp \left((\xi_i \ \xi_j) \begin{pmatrix} T^{-1} & 0 \\ 1-T^{-1} & 1 \end{pmatrix} \begin{pmatrix} x_k \\ x_l \end{pmatrix} \right) \\ P_{ij}[T] &\mapsto \exp(\xi_i x_j) \end{aligned}$$

with

(Note that the matrices appearing in these formulas are the Burau matrices).

3. An Implementation of \mathcal{A}

If I didn't implement I wouldn't believe myself.

Written in Mathematica [Wo], available as the notebook Alpha.nb at <http://drorbn.net/mo21/ap>. Code lines are highlighted in grey, demo lines are plain. We start with an implementation of elements ("Wedge") of exterior algebras, and of the wedge product ("WP"):

```
WP[Wedge[u___], Wedge[v___]] := Signature[{u, v}] * Wedge @@ Sort[{u, v}];
WP[0, _] = WP[_, 0] = 0;
WP[A_, B_] :=
  Expand[Distribute[A ** B] /.
    (a_. * u_Wedge) ** (b_. * v_Wedge) -> a b WP[u, v]];
WP[Wedge[a_] + Wedge[b] - 2 b a, Wedge[a] - 3 Wedge[b] + 7 c a d]
Wedge[] + Wedge[a] - 3 Wedge[b] - a b + 7 c a d + 7 a c a d + 14 a b a c a d
```

Comments.

- ▶ We can relax $|\mathcal{X}| = |X|$ at no cost.
- ▶ We can lose the distinction between \mathcal{X} and X and get "circuit algebras".
- ▶ There is a "coloured version", where $\mathcal{T}(\mathcal{X}, X)$ is replaced with $\mathcal{T}(\mathcal{X}, X, \lambda, l)$ where $\lambda : \mathcal{X} \rightarrow C$ and $l : X \rightarrow C$ are "colour functions" into some set C of "colours", and contractions $c_{x,\xi}$ are allowed only if x and ξ are of the same colour, $l(x) = \lambda(\xi)$. In the world of tangles, this is "coloured tangles".

Alternative Formulations.

- ▶ $c_{x,\xi} w = \iota_{\xi} \iota_x e^{x\xi} w$, where ι_{\cdot} denotes interior multiplication.
- ▶ Using Fermionic integration, $c_{x,\xi} w = \int e^{x\xi} w d\xi dx$.
- ▶ $c_{x,\xi}$ represents composition in exterior algebras! With $X^* := \{x^* : x \in X\}$, we have that $\text{Hom}(\Lambda X, \Lambda Y) \cong \Lambda(X^* \sqcup Y)$ and the following square commutes:

$$\begin{array}{ccc} \text{Hom}(\Lambda X, \Lambda Y) \otimes \text{Hom}(\Lambda Y, \Lambda Z) & \xrightarrow{\parallel} & \text{Hom}(\Lambda X, \Lambda Z) \\ \uparrow & & \uparrow \\ \Lambda(X^* \sqcup Y \sqcup Y^* \sqcup Z) & \xrightarrow{\prod_{y \in Y} c_{y,y^*}} & \Lambda(X^*, Z) \end{array}$$

- ▶ Similarly, $\Lambda(\mathcal{X} \sqcup X) \cong (H^*)^{\otimes \mathcal{X}} \otimes H^{\otimes X}$ where H is a 2-dimensional "state space" and H^* is its dual. Under this identification, $c_{x,\xi}$ becomes the contraction of an H factor with an H^* factor.

Theorem.

If D is a classical link diagram with k components coloured T_1, \dots, T_k whose first component is open and the rest are closed, if MVA is the multivariable Alexander polynomial of the closure of D (with these colours), and if p_j is the counterclockwise rotation number of the j th component of D , then

$$\mathcal{A}(D) = T_1^{-1/2} (T_1 - 1) \left(\prod_j T_j^{p_j/2} \right) \cdot MVA \cdot (1 + \xi_{in} \wedge x_{out}).$$

(\mathcal{A} vanishes on closed links).