

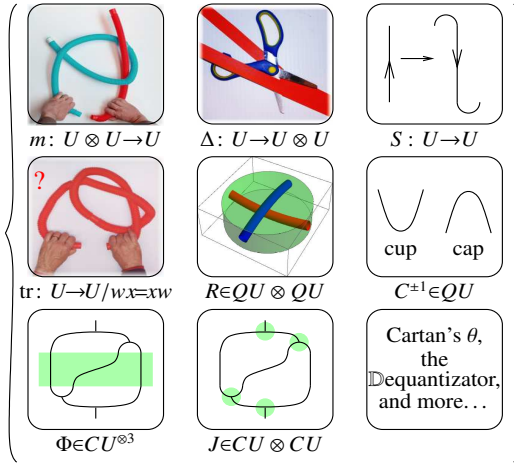


# Everything around $sl_{2+}^\epsilon$ is DoPeGDO. So what?

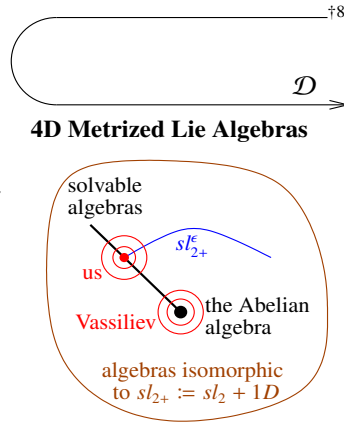
**Abstract.** I'll explain what "everything around" means: classical and quantum  $m, \Delta, S, tr, R, C$ , and  $\theta$ , as well as  $P, \Phi, J, \mathbb{D}$ , and more, and all of their compositions. What **DoPeGDO** means: the category of **Docile Perturbed Gaussian Differential Operators**. And what  $sl_{2+}^\epsilon$  means: a solvable approximation of the simple Lie algebra  $sl_2$ .

Knot theorists should rejoice because all this leads to very powerful and well-behaved poly-time-computable knot invariants. Quantum algebraists should rejoice because it's a realistic playground for testing complicated equations and theories.

**Conventions.** 1. For a set  $A$ , let  $z_A := \{z_i\}_{i \in A}$  and let  $\zeta_A := \{\zeta_i^* = \zeta_i\}_{i \in A}$ .<sup>†1</sup> 2. Everything converges!



## Less Abstract



**DoPeGDO** := The category with objects finite sets<sup>†2</sup> and  $\text{mor}(A \rightarrow B)$ :

$$\{\mathcal{F} = \omega \exp(Q + P)\} \subset \mathbb{Q}[\zeta_A, z_B, \epsilon]$$

Where: •  $\omega$  is a scalar.<sup>†3</sup> •  $Q$  is a "small"  $\epsilon$ -free quadratic in  $\zeta_A \cup z_B$ .<sup>†4</sup> •  $P$  is a "docile perturbation":  $P = \sum_{k \geq 1} \epsilon^k P^{(k)}$ , where  $\text{deg } P^{(k)} \leq 2k + 2$ .<sup>†5</sup> • Compositions:<sup>†6</sup>

$$\mathcal{F} // \mathcal{G} = \mathcal{G} \circ \mathcal{F} := (\mathcal{G}|_{\zeta_i \rightarrow \partial_{\zeta_i} \mathcal{F}})_{z_i=0} = (\mathcal{F}|_{z_i \rightarrow \partial_{\zeta_i} \mathcal{G}})_{\zeta_i=0}$$

**Cool!**  $(V^*)^{\otimes \Sigma} \otimes V^{\otimes \Sigma}$  explodes; the ranks of quadratics and bounded-degree polynomials grow slowly!<sup>†7</sup> **Representation theory is over-rated!**

**Cool!** How often do you see a computational toolbox so successful?

**Our Algebras.** Let  $sl_{2+}^\epsilon := L\langle y, b, a, x \rangle$  subject to  $[a, x] = x$ ,  $[b, y] = -\epsilon y$ ,  $[a, b] = 0$ ,  $[a, y] = -y$ ,  $[b, x] = \epsilon x$ , and  $[x, y] = \epsilon a + b$ . So  $t := \epsilon a - b$  is central and if  $\exists \epsilon^{-1}$ ,  $sl_{2+}^\epsilon / \langle t \rangle \cong sl_2$ .<sup>ωεβ/oa</sup>  $U$  is either  $CU = \mathcal{U}(sl_{2+}^\epsilon)[[\hbar]]$  or  $QU = \mathcal{U}_\hbar(sl_{2+}^\epsilon) = A\langle y, b, a, x \rangle[[\hbar]]$  with  $[a, x] = x$ ,  $[b, y] = -\epsilon y$ ,  $[a, b] = 0$ ,  $[a, y] = -y$ ,  $[b, x] = \epsilon x$ , and  $xy - qyx = (1 - AB)/\hbar$ , where  $q = e^{\hbar \epsilon}$ ,  $A = e^{-\hbar \epsilon a}$ , and  $B = e^{-\hbar b}$ . Set also  $T = A^{-1}B = e^{\hbar t}$ .

**The Quantum Leap.** Also decree that in  $QU$ ,

$$\Delta(y, b, a, x) = (y_1 + B_1 y_2, b_1 + b_2, a_1 + a_2, x_1 + A_1 x_2),$$
$$S(y, b, a, x) = (-B^{-1} y, -b, -a, -A^{-1} x),$$

and  $R = \sum \hbar^{j+k} y^k b^j \otimes a^j x^k / j! [k]_q!$ .

**Mid-Talk Debts.** • What is this good for in quantum algebra?

- In knot theory?
- How does the "inclusion"  $\mathcal{D}: \text{Hom}(U^{\otimes \Sigma} \rightarrow U^{\otimes S}) \rightsquigarrow$  **DoPeGDO** work?
- Proofs that everything around  $sl_{2+}^\epsilon$  really is **DoPeGDO**.
- Relations with prior art.
- The rest of the "compositions" story.

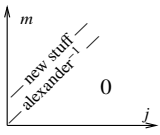
**Theorem** ([BG], conjectured [MM], elucidated [Ro1]). Let  $J_d(K)$  be the coloured Jones polynomial of  $K$ , in the  $d$ -dimensional representation of  $sl_2$ . Writing

$$\left. \frac{(q^{1/2} - q^{-1/2}) J_d(K)}{q^{d/2} - q^{-d/2}} \right|_{q=e^\hbar} = \sum_{j,m \geq 0} a_{jm}(K) d^j \hbar^m,$$

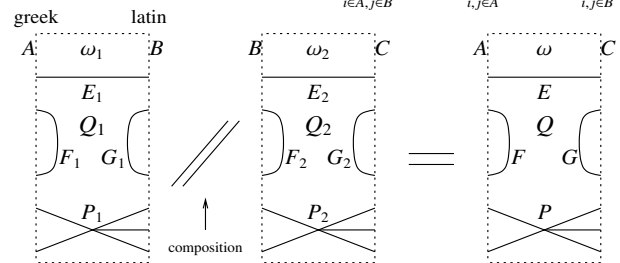
"below diagonal" coefficients vanish,  $a_{jm}(K) = 0$  if  $j > m$ , and "on diagonal" coefficients give the inverse of the Alexander polynomial:  $(\sum_{m=0}^\infty a_{mm}(K) \hbar^m) \cdot \omega(K)(e^\hbar) = 1$ .

"Above diagonal" we have **Rozansky's Theorem** [Ro3, (1.2)]:

$$J_d(K)(q) = \frac{q^d - q^{-d}}{(q - q^{-1}) \omega(K)(q^d)} \left( 1 + \sum_{k=1}^\infty \frac{(q-1)^k \rho_k(K)(q^d)}{\omega^{2k}(K)(q^d)} \right).$$



**Compositions (1).** In  $\text{mor}(A \rightarrow B)$ ,  $Q = \sum_{i \in A, j \in B} E_{ij} \zeta_i z_j + \frac{1}{2} \sum_{i, j \in A} F_{ij} \zeta_i \zeta_j + \frac{1}{2} \sum_{i, j \in B} G_{ij} z_i z_j$



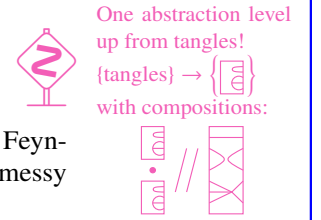
Where •  $E = E_1(I - F_2 G_1)^{-1} E_2$ .

•  $F = F_1 + E_1 F_2 (I - G_1 F_2)^{-1} E_1^T$ .

•  $G = G_2 + E_2^T G_1 (I - F_2 G_1)^{-1} E_2$ .

•  $\omega = \omega_1 \omega_2 \det(I - F_2 G_1)^{-1}$ .

•  $P$  is computed using "connected Feynman diagrams" or as the solution of a messy PDE (yet we're still in algebra!).



**DoPeGDO Footnotes.** †1. Each variable has a "weight"  $\in \{0, 1, 2\}$ , and always  $\text{wt } z_i + \text{wt } \zeta_i = 2$ .

†2. Really, "weight-graded finite sets"  $A = A_0 \sqcup A_1 \sqcup A_2$ .

†3. Really, a power series in the weight-0 variables<sup>†9</sup>.

†4. The weight of  $Q$  must be 2, so it decomposes as  $Q = Q_{20} + Q_{11}$ . The coefficients of  $Q_{20}$  are rational numbers while the coefficients of  $Q_{11}$  may be weight-0 power series<sup>†9</sup>.

†5. Setting  $\text{wt } \epsilon = -2$ , the weight of  $P$  is  $\leq 2$  (so the powers of the weight-0 variables are not constrained<sup>†9</sup>).

†6. There's also an obvious product

$$\text{mor}(A_1 \rightarrow B_1) \times \text{mor}(A_2 \rightarrow B_2) \rightarrow \text{mor}(A_1 \sqcup A_2 \rightarrow B_1 \sqcup B_2).$$

†7. That is, if the weight-0 variables are ignored. Otherwise more care is needed yet the conclusion remains.

†8.  $\text{Hom}(U^{\otimes \Sigma} \rightarrow U^{\otimes S}) \rightsquigarrow \text{mor}(\{\eta_i, \beta_i, \tau_i, \alpha_i, \xi_i\}_{i \in \Sigma} \rightarrow \{y_i, b_i, t_i, a_i, x_i\}_{i \in S})$ , where  $\text{wt}(\eta_i, \xi_i, y_i, x_i) = 1$  and  $\text{wt}(\beta_i, \tau_i, \alpha_i; b_i, t_i, a_i) = (2, 2, 0; 0, 0, 2)$ .

†9. For tangle invariants the wt-0 power series are always rational functions in the exponentials of the wt-0 variables (for knots: just one variable), with degrees bounded linearly by the crossing number.