$\mathcal{D}\colon \operatorname{Hom}(U^{\otimes\Sigma}\to U^{\otimes S})\to\mathbb{Q}[\![\eta_\Sigma,\beta_\Sigma,\alpha_\Sigma,\xi_\Sigma,y_S,b_S,a_S,x_S]\!]$. The PBW theorem for CU (always in the *ybax* order), or its quantum analog for QU, say that if $U=CU$ or QU then $U^{\otimes S}$ is isomorphic as a vector space to $\mathbb{Q}[y_i,b_i,a_i,x_i]_{i\in S}[\![\hbar]\!]$; so it is enough to understand $\operatorname{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B])$ for finite sets A and B.

Claim. $F\in\operatorname{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B]) \xrightarrow[\mathcal{D}]{\sim} \mathbb{Q}[z_B][\![\zeta_A]\!]\ni\mathcal{F}$ via

$$\mathcal{D}(F):=\sum_{n\in\mathbb{N}^A}\frac{\zeta_A^n}{n!}F(z_A^n)=F\left(\mathbb{e}^{\sum_{a\in A}\zeta_a z_a}\right)=\mathcal{F},$$

$$\mathcal{D}^{-1}(\mathcal{F})(p)=\left(p|_{z_a\to\partial_{\zeta_a}}\mathcal{F}\right)_{\zeta_a=0}\qquad\text{for }p\in\mathbb{Q}[z_A].$$

Claim. Assuming convergence, if $F\in\operatorname{Hom}(\mathbb{Q}[z_A]\to\mathbb{Q}[z_B])$, $G\in\operatorname{Hom}(\mathbb{Q}[z_B]\to\mathbb{Q}[z_C])$, $\mathcal{F}=\mathcal{D}(F)$, and $\mathcal{G}=\mathcal{D}(G)$, then

$$\mathcal{D}(F/\!\!/G)=\left(\mathcal{F}|_{z_i\to\partial_{\zeta_i}}\mathcal{G}\right)_{\zeta_i=0}.$$

And so the title of the talk finally makes sense!

Example. $\mathcal{D}(id\colon U\to U)=\mathbb{e}^{\eta y+\beta b+\alpha a+\xi x}$.

Example. Let $c\Delta^i_{jk}\colon CU^{\otimes\{i\}}\to CU^{\otimes\{j,k\}}$ be the standard coproduct, given by $c\Delta^i_{jk}(y_i,b_i,a_i,x_i)=(y_j+y_k,b_j+b_k,a_j+a_k,x_j+x_k)$. Then

$$\mathcal{D}(c\Delta^i_{jk})=c\Delta^i_{jk}(\mathbb{e}^{\eta_iy_i+\beta_ib_i+\alpha_ia_i+\xi_ix_i})$$
$$=\mathbb{e}^{\eta_i(y_j+y_k)+\beta_i(b_j+b_k)+\alpha_i(a_j+a_k)+\xi_i(x_j+x_k)}.$$

Example. The standard commutative product m^{ij}_k of polynomials is given by $z_i,z_j\to z_k$. Hence $\mathcal{D}(m^{ij}_k)=m^{ij}_k(\mathbb{e}^{\zeta_iz_i+\zeta_jz_j})=\mathbb{e}^{(\zeta_i+\zeta_j)z_k}$.

$$\begin{array}{ccc}\mathbb{Q}[z]_i\otimes\mathbb{Q}[z]_j & \xrightarrow{m^{ij}_k} & \mathbb{Q}[z]_k\\ \| & & \|\\ \mathbb{Q}[z_i,z_j] & \xrightarrow{m^{ij}_k} & \mathbb{Q}[z_k]\end{array}$$

A real DoPeGDO Example. Let $cm^{ij}_k\colon CU_i\otimes CU_j\to CU_k$ be "classical multiplication" for sl^ϵ_{2+}, and let $\mathbb{O}_i\colon\mathbb{Q}[y_i,b_i,a_i,x_i]\to CU_i$ be the PBW ordering map.

$$\begin{array}{ccc}CU_i\otimes CU_j & \xrightarrow{cm^{ij}_k} & CU_k\\ \uparrow{\scriptstyle\mathbb{O}_{i,j}} & & \uparrow{\scriptstyle\mathbb{O}_k}\\ \mathbb{Q}[y_i,b_i,a_i,x_i,y_j,b_j,a_j,x_j] & & \mathbb{Q}[y_k,b_k,a_k,x_k]\end{array}$$

Claim. Let (all brawn and no brains)

$$\Lambda=\left(\eta_i+\frac{e^{-\alpha_i-\epsilon\beta_i}\eta_j}{1+\epsilon\eta_j\xi_i}\right)y_k+\left(\beta_i+\beta_j+\frac{\log\left(1+\epsilon\eta_j\xi_i\right)}{\epsilon}\right)b_k+$$
$$\left(\alpha_i+\alpha_j+\log\left(1+\epsilon\eta_j\xi_i\right)\right)a_k+\left(\frac{e^{-\alpha_j-\epsilon\beta_j}\xi_i}{1+\epsilon\eta_j\xi_i}+\xi_j\right)x_k$$

Then $\mathbb{e}^{\eta_iy_i+\beta_ib_i+\alpha_ia_i+\xi_ix_i+\eta_jy_j+\beta_jb_j+\alpha_ja_j+\xi_jx_j}/\!\!/\mathbb{O}_{i,j}/\!\!/cm^{ij}_k=\mathbb{e}^{\Lambda}/\!\!/\mathbb{O}_k$, and hence $\mathcal{D}(cm^{ij}_k)=\mathbb{e}^\Lambda$ and cm^{ij}_k is DoPeGDO.

Proof. We compute in a faithful 2D representation $z\mapsto\hat{z}$ of CU: (ωεβ/cm)

```
HL[ℰ_] := Style[ℰ, Background → If[TrueQ@ℰ, ■, ■]];
{ŷ = (0 0
      ε 0), b̂ = (0  0
                 0 -ε), â = (1 0
                             0 0), x̂ = (0 1
                                        0 0)};
HL /@ {â.x̂ - x̂.â == x̂, â.ŷ - ŷ.â == -ŷ, b̂.ŷ - ŷ.b̂ == -ε ŷ,
  b̂.x̂ - x̂.b̂ == ε x̂, x̂.ŷ - ŷ.x̂ == b̂ + ε â}

{True, True, True, True, True}
HL@Simplify@With[{𝔼 = MatrixExp},
   𝔼[ηi ŷ].𝔼[βi b̂].𝔼[αi â].𝔼[ξi x̂].𝔼[ηj ŷ].𝔼[βj b̂].
    𝔼[αj â].𝔼[ξj x̂] == 𝔼[ŷ ∂yk Λ].𝔼[b̂ ∂bk Λ].𝔼[â ∂ak Λ].
    𝔼[x̂ ∂xk Λ]]

True
Series[Λ, {ϵ, 0, 1}]
(ak (αi + αj) + yk (ηi + e^-αi ηj) +
   bk (βi + βj + ηj ξi) + xk (e^-αj ξi + ξj)) +
 (ak ηj ξi - 1/2 bk ηj^2 ξi^2 - e^-αi yk ηj (βi + ηj ξi) -
   e^-αj xk ξi (βj + ηj ξi)) ϵ + O[ϵ]^2
```

(Shame, but this technique fails for QU).

Claim. In QU, R is DoPeGDO.

Proof. Recall that with $q=\mathbb{e}^{\hbar\epsilon}$,

$$R=\sum\hbar^{j+k}y^kb^j\otimes a^jx^k/j![k]_q!=\mathbb{O}\left(\mathbb{e}^{\hbar b_1a_2}\mathbb{e}_q^{\hbar y_1x_2}\right).$$

Now expand $\mathbb{e}_q^{\hbar y_1x_2}$ in powers of ϵ using:

Faddeev's Formula (In as much as we can tell, first appeared without proof in Faddeev [Fa], rediscovered and proven in Quesne [Qu], and again with easier proof, in Zagier [Za]). With $[n]_q:=\frac{q^n-1}{q-1}$, with $[n]_q!:=[1]_q[2]_q\cdots[n]_q$ and with $\mathbb{e}_q^x:=\sum_{n\ge0}\frac{x^n}{[n]_q!}$, we have

$$\log\mathbb{e}_q^x=\sum_{k\ge1}\frac{(1-q)^kx^k}{k(1-q^k)}=x+\frac{(1-q)^2x^2}{2(1-q^2)}+\dots.$$

Proof. We have that $\mathbb{e}_q^x=\frac{\mathbb{e}_q^{qx}-\mathbb{e}_q^x}{qx-x}$ ("the q-derivative of $\mathbb{e}_q^x$ is itself"), and hence $\mathbb{e}_q^{qx}=(1+(1-q)x)\mathbb{e}_q^x$, and

$$\log\mathbb{e}_q^{qx}=\log(1+(1-q)x)+\log\mathbb{e}_q^x.$$

Writing $\log\mathbb{e}_q^x=\sum_{k\ge1}a_kx^k$ and comparing powers of x, we get $q^ka_k=-(1-q)^k/k+a_k$, or $a_k=\frac{(1-q)^k}{k(1-q^k)}$. □

Compositions (2). Recall that with all indices i running in some set B,

$$\mathcal{F}/\!\!/\mathcal{G}=\left(\mathcal{F}|_{z_i\to\partial_{\zeta_i}}\mathcal{G}\right)_{\zeta_i=0}\overset{(1)}{=}\mathbb{e}^{\sum\partial_{z_i}\partial_{\zeta_i}}(\mathcal{F}\mathcal{G})\Big|_{z_i=\zeta_i=0},$$

(1) Strictly speaking, true only when $B\cap(A\cup C)=\emptyset$.

so in general we wish to understand

$$[F\colon\mathcal{E}]_B:=\mathbb{e}^{\frac12\sum_{i,j\in B}F_{ij}\partial_{z_i}\partial_{z_j}}\mathcal{E}\quad\text{and}\quad\langle F\colon\mathcal{E}\rangle_B:=[F\colon\mathcal{E}]_B|_{z_B\to0},$$

where $\mathcal{E}$ is a docile perturbed Gaussian. The following lemma allows us to restrict to the case where $\mathcal{E}$ has no B-B quadratic part:

Lemma 1. With convergences left to the reader,

$$\left\langle F\colon\mathcal{E}\mathbb{e}^{\frac12\sum_{i,j\in B}G_{ij}z_iz_j}\right\rangle_B=\det(1-GF)^{-1/2}\left\langle F(1-GF)^{-1}\colon\mathcal{E}\right\rangle_B.$$

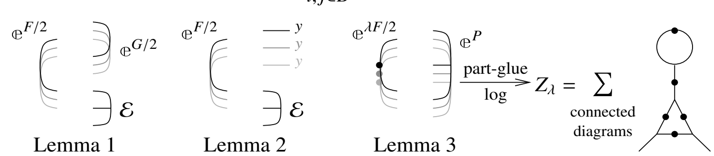
The next lemma dispatches the case where $\mathcal{E}$ has a B-linear part:

Lemma 2. $\left\langle F\colon\mathcal{E}\mathbb{e}^{\sum_{i\in B}y_iz_i}\right\rangle_B=\mathbb{e}^{\frac12\sum_{i,j\in B}F_{ij}y_iy_j}\left\langle F\colon\ \mathcal{E}|_{z_B\to z_B+Fy_B}\right\rangle_B$.

Finally, we deal with the docile perturbation case:

Lemma 3. With an extra variable λ, $Z_\lambda:=\log[\lambda F\colon\mathbb{e}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0=P\quad\text{and}\quad\partial_\lambda Z_\lambda=\frac12\sum_{i,j\in B}F_{ij}\left(\partial_{z_i}\partial_{z_j}Z_\lambda+(\partial_{z_i}Z_\lambda)(\partial_{z_j}Z_\lambda)\right).$$



Complexity to ϵ^k, for an n-xing width w knot (by [LT], $w\in O(\sqrt{n})$), is $O(n^2w^{2k+2}\log n)=O(n^{k+3}\log n)$ integer operations.