Define[lhs = rhs, ...] defines the lhs to be rhs, except that rhs is computed only once for each value of \$k. Fancy Mathematica not for the faint of heart. Most readers should ignore.

```
SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs];);
Define[op_is__ = ℰ_] :=
 Module[{SD, ii, jj, kk, isp, nis, nisp, sis},
  Block[{i, j, k},
   ReleaseHold[Hold[
      SD[op_{nisp,$k_Integer}, PP_Boot@Block[{i, j, k}, op_{isp,$k} = ℰ;
         op_{nis,$k}]];
      SD[op_isp, op_{{is},$k}]; SD[op_{sis__}, op_{{sis}}];
     ] /. {SD → SetDelayed,
      isp → {is} /. {i → i_, j → j_, k → k_},
      nis → {is} /. {i → ii, j → jj, k → kk},
      nisp → {is} /. {i → ii_, j → jj_, k → kk_}
     }] ]]
```

The Objects

ωεβ/objects

Symmetric Algebra Objects

```
sm_{i_,j_→k_} :=
  𝔼_{{i,j}→{k}}[b_k (β_i + β_j) + t_k (τ_i + τ_j) + a_k (α_i + α_j) +
    y_k (η_i + η_j) + x_k (ξ_i + ξ_j)];
sΔ_{i_→j_,k_} :=
  𝔼_{{i}→{j,k}}[β_i (b_j + b_k) + τ_i (t_j + t_k) + α_i (a_j + a_k) +
    η_i (y_j + y_k) + ξ_i (x_j + x_k)];
sS_i_ := 𝔼_{{i}→{i}}[-β_i b_i - τ_i t_i - α_i a_i - η_i y_i - ξ_i x_i];
sϵ_i_ := 𝔼_{{}→{i}}[0];
sη_i_ := 𝔼_{{i}→{}}[0];
sσ_{i_→j_} := 𝔼_{{i}→{j}}[β_i b_j + τ_i t_j + α_i a_j + η_i y_j + ξ_i x_j];
sΥ_{i_→j_,k_,l_,m_} := 𝔼_{{i}→{j,k,l,m}}[β_i b_k + τ_i t_k + α_i a_l + η_i y_j + ξ_i x_m];
```

The CU Definitions

```
cΛ = (η_i + e^{-γ α_i - ϵ β_i} η_j / (1 + γ ϵ η_j ξ_i)) y_k + (β_i + β_j + Log[1 + γ ϵ η_j ξ_i]/ϵ) b_k +
   (α_i + α_j + Log[1 + γ ϵ η_j ξ_i]/γ) a_k + (e^{-γ α_j - ϵ β_j} ξ_i / (1 + γ ϵ η_j ξ_i) + ξ_j) x_k;
Define[cm_{i,j→k} = 𝔼_{{i,j}→{k}}[cΛ]]
Define[cσ_{i→j} = sσ_{i,j} /. τ_i → 0, cϵ_i = sϵ_i, cη_i = sη_i,
  cΔ_{i→j,k} = sΔ_{i→j,k},
  cS_i = sS_i // sΥ_{i→1,2,3,4} // cm_{4,3→i} // cm_{i,2→i} // cm_{i,1→i}];
```

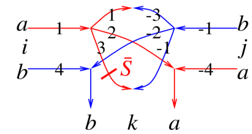
Booting Up QU

```
Define[aσ_{i→j} = 𝔼_{{i}→{j}}[a_j α_i + x_j ξ_i],
 bσ_{i→j} = 𝔼_{{i}→{j}}[b_j β_i + y_j η_i]]
Define[am_{i,j→k} = 𝔼_{{i,j}→{k}}[(α_i + α_j) a_k + (𝒜_j^{-1} ξ_i + ξ_j) x_k],
 bm_{i,j→k} = 𝔼_{{i,j}→{k}}[(β_i + β_j) b_k + (η_i + e^{-ϵ β_i} η_j) y_k]]
Define[R_{i,j} = 𝔼_{{}→{i,j}}[ℏ a_j b_i + Sum_{k=1}^{$k+1} (1 - e^{γ ϵ ℏ})^k (ℏ y_i x_j)^k / (k (1 - e^{k γ ϵ ℏ}))],
 R̄_{i,j} = CF@𝔼_{{}→{i,j}}[-ℏ a_j b_i, -ℏ x_j y_i / B_i,
    1 + If[$k == 0, 0, (R̄_{{i,j},$k-1})_{$k}[3] -
      (((R̄_{{i,j},0})_{$k} R_{1,2} (R̄_{{3,4},$k-1})_{$k}) // (bm_{i,1→i} am_{j,2→j}) //
        (bm_{i,3→i} am_{j,4→j}))[3]]],
 P_{i,j} = 𝔼_{{i,j}→{}}[β_i α_j / ℏ, η_i ξ_j / ℏ,
   1 + If[$k == 0, 0, (P_{{i,j},$k-1})_{$k}[3] -
     (R_{1,2} // ((P_{{1,j},0})_{$k} (P_{{i,2},$k-1})_{$k}))[3]]]]
```

```
Define[aS_i = (aσ_{i→2} R̄_{1,i}) // P_{1,2},
 aS̄_i = 𝔼_{{i}→{i}}[-a_i α_i, -x_i 𝒜_i ξ_i,
   1 + If[$k == 0, 0, (aS̄_{{i},$k-1})_{$k}[3] -
     ((aS̄_{{i},0})_{$k} // aS_i // (aS̄_{{i},$k-1})_{$k})[3]]]]
```

```
Define[bS_i = bσ_{i→1} R_{i,2} // aS_2 // P_{1,2},
 bS̄_i = bσ_{i→1} R_{i,2} // aS̄_2 // P_{1,2},
 aΔ_{i→j,k} = (R_{1,j} R_{2,k}) // bm_{1,2→3} // P_{3,i},
 bΔ_{i→j,k} = (R_{j,1} R_{k,2}) // am_{1,2→3} // P_{i,3}]
```

The Drinfel'd double:




```
Define[
 dm_{i,j→k} =
  ((sΥ_{i→4,4,1,1} // aΔ_{1→1,2} // aΔ_{2→2,3} // aS̄_3)
    (sΥ_{j→-1,-1,-4,-4} // bΔ_{-1→-1,-2} // bΔ_{-2→-2,-3})) //
   (P_{-1,3} P_{-3,1} am_{2,-4→k} bm_{4,-2→k})]
```

```
Define[dσ_{i→j} = aσ_{i→j} bσ_{i→j},
 dϵ_i = sϵ_i, dη_i = sη_i,
 dS_i = sΥ_{i→1,1,2,2} // (bS̄_1 aS_2) // dm_{2,1→i},
 dS̄_i = sΥ_{i→1,1,2,2} // (bS_1 aS̄_2) // dm_{2,1→i},
 dΔ_{i→j,k} = (bΔ_{i→3,1} aΔ_{i→2,4}) // (dm_{3,4→k} dm_{1,2→j})]
```

```
Define[C_i = 𝔼_{{}→{i}}[0, 0, B_i^{1/2} e^{-ℏ ϵ a_i/2}]_{$k},
 C̄_i = 𝔼_{{}→{i}}[0, 0, B_i^{-1/2} e^{ℏ ϵ a_i/2}]_{$k},
 Kink_i = (R_{1,3} C̄_2) // dm_{1,2→1} // dm_{1,3→i},
 K̄ink_i = (R̄_{1,3} C_2) // dm_{1,2→1} // dm_{1,3→i}]
```

Note. $t == \epsilon a - \gamma b$ and $b == -t/\gamma + \epsilon a/\gamma$.

```
Define[b2t_i = 𝔼_{{i}→{i}}[α_i a_i + β_i (ϵ a_i - t_i) / γ + ξ_i x_i + η_i y_i],
 t2b_i = 𝔼_{{i}→{i}}[α_i a_i + τ_i (ϵ a_i - γ b_i) + ξ_i x_i + η_i y_i]]
```

The Knot Tensors

```
Define[kR_{i,j} = R_{i,j} // (b2t_i b2t_j) /. t_{i|j} → t,
 kR̄_{i,j} = R̄_{i,j} // (b2t_i b2t_j) /. {t_{i|j} → t, T_{i|j} → T},
 km_{i,j→k} = (t2b_i t2b_j) // dm_{i,j→k} //
   b2t_k /. {t_k → t, T_k → T, τ_{i|j} → 0},
 kC_i = C_i // b2t_i /. T_i → T,
 kC̄_i = C̄_i // b2t_i /. T_i → T,
 kKink_i = Kink_i // b2t_i /. {t_i → t, T_i → T},
 kK̄ink_i = K̄ink_i // b2t_i /. {t_i → t, T_i → T}]
```

Some of the Atoms.

ωεβ/atoms

With $\mathcal{A}_i := e^{\alpha_i}$ and $B_i = e^{-b_i}$,

```
PP_ := Identity; $k = 1; ℏ = γ = 1;
Column[
 (# → (ℰ = ToExpression[#];
      Normal@Simplify[ℰ⟦1⟧ + ℰ⟦2⟧ + Log@ℰ⟦3⟧])) & /@
  {"dm_{i,j→k}", "dΔ_{i→j,k}", "dS_i", "R_{i,j}", "P_{i,j}"}]
```

Video and more: http://www.math.toronto.edu/~drorbn/Talks/CRM-1907, http://www.math.toronto.edu/~drorbn/Talks/UCLA-191101.