

The Real Thing. In the algebra QU_ϵ , over $\mathbb{Q}[[\hbar]]$ using the $yaxt$ order, $T = e^{\hbar t}$, $\tilde{T} = T^{-1}$, $\mathcal{A} = e^\alpha$, and $\tilde{\mathcal{A}} = \mathcal{A}^{-1}$, we have

$$\tilde{R}_{ij} = e^{\hbar(y_i x_j - t_i a_j)} \left(1 + \epsilon \hbar (a_i a_j - \hbar^2 y_i^2 x_j^2 / 4) + O(\epsilon^2) \right)$$

in $\mathcal{S}(B_i, B_j)$, and in $\mathcal{S}(B_1^*, B_2^*, B)$ we have

$$\tilde{m} = e^{(\alpha_1 + \alpha_2) a + \eta_2 \xi_1 (1-T)/\hbar + (\xi_1 \tilde{\mathcal{A}}_2 + \xi_2) x + (\eta_1 + \eta_2 \tilde{\mathcal{A}}_1) y} \left(1 + \epsilon \lambda + O(\epsilon^2) \right),$$

where $\lambda = \frac{2a\eta_2 \xi_1 T + \eta_2^2 \xi_1^2 (3T^2 - 4T + 1)/4\hbar - \eta_2 \xi_1^2 (3T - 1) x \tilde{\mathcal{A}}_2 / 2 - \eta_2^2 \xi_1 (3T - 1) y \tilde{\mathcal{A}}_1 / 2 + \eta_2 \xi_1 x y \hbar \tilde{\mathcal{A}}_1 \tilde{\mathcal{A}}_2}{}$.

Finally,

$$\tilde{\Delta} = e^{\tau(t_1 + t_2) + \eta(y_1 + T_1 y_2) + \alpha(a_1 + a_2) + \xi(x_1 + x_2)} (1 + O(\epsilon)) \in \mathcal{S}(B^*, B_1, B_2),$$

$$\text{and } \tilde{S} = e^{-\tau t - \alpha a - \eta \xi (1 - \tilde{T}) \mathcal{A} / \hbar - \tilde{T} \eta y \mathcal{A} - \xi x \mathcal{A}} (1 + O(\epsilon)) \in \mathcal{S}(B^*, B).$$

The Zipping Issue.

(between unbound and bound lies half-zipped).



Zipping. If $P(\zeta^j, z_i)$ is a polynomial, or whenever otherwise convergent, set $\langle P(\zeta^j, z_i) \rangle_{(\zeta^j)} = P(\partial_{z_j}, z_i) \Big|_{z_i=0}$. (E.g., if $P = \sum a_{nm} \zeta^n z^m$ then $\langle P \rangle_\zeta = \sum a_{nm} \partial_z^n z^m \Big|_{z=0} = \sum n! a_{nm}$).

The Zipping / Contraction Theorem. If $P = P(\zeta^j, z_i)$ has a finite ζ -degree and the y 's and the q 's are "small" then

$$\left\langle P e^{c + \eta^j z_j + y_i \zeta^i + q_j^i z_i \zeta^j} \right\rangle_{(\zeta^j)} = \det(\tilde{q}) e^{c + \eta^j q_j^k} \left\langle P \Big|_{z_i \rightarrow \tilde{q}_i^k (z_k + y_k)} \right\rangle_{(\zeta^j)}$$

where \tilde{q} is the inverse matrix of $1 - q$: $(\delta_j^i - q_j^i) \tilde{q}_k^j = \delta_k^i$.

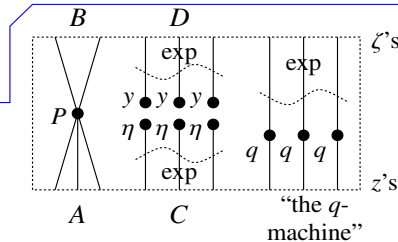
Exponential Reservoirs. The true Hilbert hotel is exp! Remove one x from an "exponential reservoir" of x 's and you are left with the same exponential reservoir:

$$e^x = \left[\dots + \frac{xxxxx}{120} + \dots \right] \xrightarrow{\partial_x} \left[\dots + \frac{xxxx}{120} + \dots \right] = (e^x)' = e^x,$$

and if you let each element choose left or right, you get twice the same reservoir:

$$e^x \xrightarrow{x \rightarrow x_l + x_r} e^{x_l + x_r} = e^{x_l} e^{x_r}.$$

A Graphical Proof. Glue top to bottom on the right, in all possible ways. Several scenarios occur:



- Start at A, go through the q -machine $k \geq 0$ times, stop at B. Get $\langle P(\zeta, \sum_{k \geq 0} q^k z) \rangle = \langle P(\zeta, \tilde{q} z) \rangle$.
- Loop through the q -machine and swallow your own tail. Get $\exp(\sum q^k / k) = \exp(-\log(1 - q)) = \tilde{q}$.
- ...

By the reservoir splitting principle, these scenarios contribute multiplicatively. □

Implementation. ($\mathbb{E}[Q, P]$ means $e^Q P$) ωεβ/Zip

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Zip_εs_list@E[Q_, P_] :=
Module[{ξ, z, zs, c, ys, ηs, qt, zrule, ξrule},
  zs = Table[ξ^*, {ξ, ξs}];
  c = Q /. Alternatives @@ (ξs ∪ zs) → 0;
  ys = Table[∂_ξ (Q /. Alternatives @@ (ξs ∪ zs) → 0), {ξ, ξs}];
  ηs = Table[∂_z (Q /. Alternatives @@ ξs → 0), {z, zs}];
  qt = Inverse@Table[Kδ_{z, z*} - ∂_{z, ε} Q, {ξ, ξs}, {z, zs}];
  zrule = Thread[zs → qt. (zs + ys)];
  ξrule = Thread[ξs → ξs + ηs.qt];
  Simplify /@
  E[c + ηs.qt.y, Det[qt] Zip_εs[P /. (zrule ∪ ξrule)]];];
```

Real Zipping is a minor mess, and is done in two phases:

	τa-phase		ξy-phase	
ζ-like variables	τ	a	ξ	y
z-like variables	t	α	x	η

Already at $\epsilon = 0$ we get the best known formulas for the Alexander polynomial!

Generic Docility. A "docile perturbed Gaussian" in the variables $(z_i)_{i \in S}$ over the ring R is an expression of the form

$$e^{q^{ij} z_i z_j} P = e^{q^{ij} z_i z_j} \left(\sum_{k \geq 0} \epsilon^k P_k \right),$$

where all coefficients are in R and where P is a "docile series": $\deg P_k \leq 4k$.

Our Docility. In the case of QU_ϵ , all invariants and operations are of the form $e^{L+Q} P$, where

- L is a quadratic of the form $\sum l_{z\zeta} z \zeta$, where z runs over $\{t_i, \alpha_i\}_{i \in S}$ and ζ over $\{\tau_i, a_i\}_{i \in S}$, with integer coefficients $l_{z\zeta}$.
- Q is a quadratic of the form $\sum q_{z\zeta} z \zeta$, where z runs over $\{x_i, \eta_i\}_{i \in S}$ and ζ over $\{\xi_i, y_i\}_{i \in S}$, with coefficients $q_{z\zeta}$ in the ring R_S of rational functions in $\{T_i, \mathcal{A}_i\}_{i \in S}$.
- P is a docile power series in $\{y_i, a_i, x_i, \eta_i, \xi_i\}_{i \in S}$ with coefficients in R_S , and where $\deg(y_i, a_i, x_i, \eta_i, \xi_i) = (1, 2, 1, 1, 1)$.

Docility Matters! The rank of the space of docile series to ϵ^k is polynomial in the number of variables $|S|$. !!!!

- At $\epsilon^2 = 0$ we get the Rozansky-Overbay [Ro1, Ro2, Ro3, Ov] invariant, which is stronger than HOMFLY-PT polynomial and Khovanov homology taken together!
- In general, get "higher diagonals in the Melvin-Morton-Rozansky expansion of the coloured Jones polynomial" [MM, BNG], but why spoil something good?

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"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

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