

- The “ $z_i \rightarrow z_j$  variable rename map  $\sigma_j^i: \mathcal{S}(z_i) \rightarrow \mathcal{S}(z_j)$  becomes  ${}^i\sigma_j^i = \mathbb{E}^{z_j^i}$ , and it’s easy to rename several variables simultaneously.
- The “archetypal multiplication map  $m_k^{ij}: \mathcal{S}(z_i, z_j) \rightarrow \mathcal{S}(z_k)$ ” has  ${}^i m = \mathbb{E}^{z_k(\zeta_i+\zeta_j)}$ .
- The “archetypal coproduct  $\Delta_{jk}^i: \mathcal{S}(z_i) \rightarrow \mathcal{S}(z_j, z_k)$ ”, given by  $z_i \rightarrow z_j + z_k$  or  $\Delta z = z \otimes 1 + 1 \otimes z$ , has  ${}^i \Delta = \mathbb{E}^{(z_j+z_k)\zeta_i}$ .
- $R$ -matrices tend to have terms of the form  $\mathbb{E}_q^{h y_1 x_2} \in \mathcal{U}_q \otimes \mathcal{U}_q$ . The “baby  $R$ -matrix” is  ${}^i R = \mathbb{E}^{h y x} \in \mathcal{S}(y, x)$ .

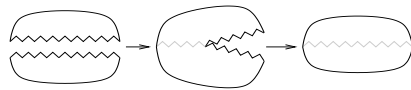
**Proposition.** If  $F: \mathcal{S}(B) \rightarrow \mathcal{S}(B')$  is linear and “continuous”, then  ${}^i F = \exp\left(\sum_{z_i \in B} \zeta_i z_i\right) // F$ .

**The Heisenberg Example.** The “Weyl form of the canonical commutation relations” states that if  $[y, x] = t$  and  $t$  is central, then  $\mathbb{E}^{\xi x} \mathbb{E}^{\eta y} = \mathbb{E}^{\eta y} \mathbb{E}^{\xi x} e^{-\eta \xi t}$ . Thus with

$$SW_{xy} \left( \begin{array}{c} \mathcal{S}(t, y, x) \\ \xrightarrow{\mathbb{O}_{xy}} \mathcal{U}(t, y, x) \\ \xleftarrow{\mathbb{O}_{yx}} \end{array} \right)$$

we have  ${}^i SW_{xy} = \mathbb{E}^{\tau t + \eta y + \xi x - \eta \xi t}$ .

**The Zipping Issue** (between unbound and bound lies half-zipped).



**Zipping.** If  $P(\zeta^j, z_i)$  is a polynomial, or whenever otherwise convergent, set

$$\left\langle P(\zeta^j, z_i) \right\rangle_{(\zeta^j)} = P\left(\partial_{z_j}, z_i\right) \Big|_{z_i=0}.$$

(E.g., if  $P = \sum a_{nm} \zeta^n z^m$  then  $\langle P \rangle_{\zeta} = \sum n! a_{nm}$ ).

**The Zipping / Contraction Theorem.** If  $P$  has a finite  $\zeta$ -degree and the  $y$ ’s and the  $q$ ’s are “small” then

$$\left\langle P(z_i, \zeta^j) \mathbb{E}^{\eta^i z_i + y_j \zeta^j} \right\rangle_{(\zeta^j)} = \left\langle P(z_i + y_i, \zeta^j) \mathbb{E}^{\eta^j (z_i + y_i)} \right\rangle_{(\zeta^j)},$$

(proof: replace  $y_j \rightarrow \hbar y_j$  and test at  $\hbar = 0$  and at  $\partial_{\hbar}$ ), and

$$\left\langle P(z_i, \zeta^j) \mathbb{E}^{c + \eta^i z_i + y_j \zeta^j + q_j^i z_i \zeta^j} \right\rangle_{(\zeta^j)} = \det(\tilde{q}) \left\langle P(\tilde{q}_i^k (z_k + y_k), \zeta^j) \mathbb{E}^{c + \eta^i \tilde{q}_i^k (z_k + y_k)} \right\rangle_{(\zeta^j)}$$

where  $\tilde{q}$  is the inverse matrix of  $1 - q$ :  $(\delta_j^i - q_j^i) \tilde{q}_k^j = \delta_k^i$  (proof: replace  $q_j^i \rightarrow \hbar q_j^i$  and test at  $\hbar = 0$  and at  $\partial_{\hbar}$ ).

**Implementation.**  $\omega \in \beta / \text{ZipBindDemo}$

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Kδ /: Kδ_{i,j} := If[i === j, 1, 0];
{z*, x*, y*} = {ξ, ε, η}; {ξ*, ε*, η*} = {z, x, y};
(u_{-i})* := (u*)_i;
Zip_{[]}[P_] := P;
Zip_{(ξ, ε, η)}[P_] :=
(Expand[P // Zip_{(ξ, ε, η)}] /. f_{-} . ξ^{d_{-}} .>> ∂_{(ξ*, ε*, η*)} f) /. ξ* → 0
Zip_{(ξ)}[(a ξ^6 + ξ + 3) (z^5 e^z + 7 z) + 99 b]
7 + 720 a + 99 b
Zip_{(ξ, η)}[ξ^3 η^3 e^{ax+by+cx}]
a^3 b^3 + 9 a^2 b^2 c + 18 a b c^2 + 6 c^3
(* E[Q,P] means e^{QP} *)
E /: Zip_{ξ, ε, η} @ E[Q_, P_] :=
Module[{ξ, z, zs, c, ys, ηs, qt, zrule, Q1, Q2},
  zs = Table[ξ*, {ξ, ξs}];
  c = Q /. Alternatives @@ (ξs ∪ zs) → 0;
  ys = Table[∂_ξ(Q /. Alternatives @@ zs → 0), {ξ, ξs}];
  ηs = Table[∂_z(Q /. Alternatives @@ ξs → 0), {z, zs}];
  qt = Inverse@Table[Kδ_{z, z*} - ∂_{z, z*} Q, {ξ, ξs}, {z, zs}];
  zrule = Thread[zs → qt. (zs + ys)];
  Q1 = c + ηs.zs /. zrule;
  Q2 = Q1 /. Alternatives @@ zs → 0;
  Simplify /@ E[Q2, Det[qt] e^{-Q2} Zip_{ξs}[e^{Q1} (P /. zrule)]];

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$$Eh = \mathbb{E} \left[ \hbar \sum_{i=1}^3 \sum_{j=1}^3 a_{10\ i+j} x_i \xi_j, \sum_{i=1}^3 f_i [x_1, x_2, x_3] \xi_i \right];$$

$$E1 = Eh /. \hbar \rightarrow 1$$

$$\mathbb{E} [a_{11} x_1 \xi_1 + a_{21} x_2 \xi_1 + a_{31} x_3 \xi_1 + a_{12} x_1 \xi_2 + a_{22} x_2 \xi_2 + a_{32} x_3 \xi_2 + a_{13} x_1 \xi_3 + a_{23} x_2 \xi_3 + a_{33} x_3 \xi_3, \xi_1 f_1 [x_1, x_2, x_3] + \xi_2 f_2 [x_1, x_2, x_3] + \xi_3 f_3 [x_1, x_2, x_3]]$$

$$\text{Short}[lhs = \text{Zip}_{(\xi_1, \xi_2)} @ E1, 5]$$

$$\mathbb{E} \left[ \left( (a_{13} ((-1 + a_{22}) a_{31} - a_{21} a_{32}) + a_{12} (-a_{23} a_{31} + a_{21} a_{33}) + (-1 + a_{11}) (a_{23} a_{32} - (-1 + a_{22}) a_{33})) x_3 \xi_3 \right) / (-1 + a_{12} a_{21} - a_{11} (-1 + a_{22}) + a_{22}), \frac{\llcorner 17 \gg + a_{21} \llcorner 1 \gg}{(-1 + a_{12} a_{21} - a_{11} (-1 + a_{22}) + a_{22})^2} \right]$$

$$lhs == \text{Zip}_{(\xi_1)} @ \text{Zip}_{(\xi_2)} @ E1 == \text{Zip}_{(\xi_2)} @ \text{Zip}_{(\xi_1)} @ E1$$

True

Short[

$$lhs = \text{Normal}[Eh /. \mathbb{E}[Q_, P_] \rightarrow \text{Series}[P e^Q, \{h, 0, 3\}]] // \text{Zip}_{(\xi_1, \xi_2)}, 5]$$

$$\begin{aligned} & h a_{13} \xi_3 f_1 [0, 0, x_3] + 2 h^2 a_{11} a_{13} \xi_3 f_1 [0, 0, x_3] + \\ & 3 h^3 a_{11}^2 a_{13} \xi_3 f_1 [0, 0, x_3] + 2 h^3 a_{12} a_{13} a_{21} \xi_3 f_1 [0, 0, x_3] + \\ & h^2 a_{13} a_{22} \xi_3 f_1 [0, 0, x_3] + \llcorner 337 \gg + \\ & \frac{1}{6} h^3 a_{31}^3 x_3^3 \xi_3 f_3^{(3,0,0)} [0, 0, x_3] + \frac{1}{2} h^3 a_{31}^2 a_{32} x_3^3 f_1^{(3,1,0)} [0, 0, x_3] + \\ & \frac{1}{6} h^3 a_{31}^3 x_3^3 f_2^{(3,1,0)} [0, 0, x_3] + \frac{1}{6} h^3 a_{31}^3 x_3^3 f_1^{(4,0,0)} [0, 0, x_3] \end{aligned}$$

rhs =

$$\text{Normal}[\text{Zip}_{(\xi_1, \xi_2)} @ Eh /. \mathbb{E}[Q_, P_] \rightarrow \text{Series}[P e^Q, \{h, 0, 3\}]];$$

Simplify[lhs == rhs]

True

$$E /: \mathbb{E}[Q1_, P1_] \mathbb{E}[Q2_, P2_] := \mathbb{E}[Q1 + Q2, P1 * P2];$$

$$\text{Bind}_{\xi_s, \text{List}}[L_{-E}, R_{-E}] := \text{Module}[\{n, \text{hide}\xi_s, \text{hide}z_s\},$$

$$\text{hide}\xi_s = \text{Table}[\xi_s[\text{i}] \rightarrow \xi_{\text{no}i}, \{\text{i}, \text{Length}@\xi_s\}];$$

$$\text{hide}z_s = \text{Table}[\xi_s[\text{i}]^* \rightarrow z_{\text{no}i}, \{\text{i}, \text{Length}@\xi_s\}];$$

$$\text{Zip}_{\xi_s, \text{hide}\xi_s}[L /. \text{hide}z_s](R /. \text{hide}\xi_s)];$$

$$\text{Bind}_{(\xi_2)}[\mathbb{E}[\xi(x_1 + x_2), 1], \mathbb{E}[\xi_2(x_2 + x_3), 1]]$$

$$\mathbb{E}[\xi(x_1 + x_2 + x_3), 1]$$

$$\text{Bind}_{(\xi_2)}[\mathbb{E}[(\xi_2 + \xi_3) x_2, 1], \mathbb{E}[(\xi_1 + \xi_2) x, 1]]$$

$$\mathbb{E}[x(\xi_1 + \xi_2 + \xi_3), 1]$$

**The 2D Lie Algebra.** Clever people know\* that if  $[a, x] = \gamma x$  then  $\mathbb{E}^{\xi x} \mathbb{E}^{a a} = \mathbb{E}^{a a} \mathbb{E}^{-\gamma a} \xi x$ . Ergo with

$$SW_{ax} \left( \begin{array}{c} \mathcal{S}(a, x) \\ \xrightarrow{\mathbb{O}_{ax}} \mathcal{U}(a, x) \\ \xleftarrow{\mathbb{O}_{xa}} \end{array} \right)$$

we have  ${}^i SW_{ax} = \mathbb{E}^{a a + \mathbb{E}^{-\gamma a} \xi x}$ .

\* Indeed  $xa = (a - \gamma)x$  thus  $xa^n = (a - \gamma)^n x$  thus  $x e^{a a} = e^{(a - \gamma) a} x = e^{-\gamma a} e^{a a} x$  thus  $x^n e^{a a} = e^{a a} (e^{-\gamma a})^n x^n$  thus  $\mathbb{E}^{\xi x} \mathbb{E}^{a a} = \mathbb{E}^{a a} \mathbb{E}^{-\gamma a} \xi x$ .

**The Real Thing.** In  $QU/(\epsilon^2 = 0)$  over  $\mathbb{Q}[[\hbar]]$  using the  $yax$  order,  $T = e^{\hbar t}$ ,  $\bar{T} = T^{-1}$ ,  $\mathcal{A} = e^{\gamma a}$ , and  $\bar{\mathcal{A}} = \mathcal{A}^{-1}$ , we have

$${}^i R_{ij} = \mathbb{E}^{\hbar(\gamma_i x_j - \eta_i a_j / \gamma)} \left( 1 + \epsilon \hbar (a_i a_j / \gamma - \gamma \hbar^2 y_i^2 x_j^2 / 4) \right)$$

in  $\mathcal{S}(B_i, B_j)$ , and in  $\mathcal{S}(B_1^*, B_2^*, B)$  we have

$${}^i m = \mathbb{E}^{(\alpha_1 + \alpha_2) a + \eta_2 \xi_1 (1 - T) / \hbar + (\xi_1 \bar{\mathcal{A}}_2 + \xi_2) x + (\eta_1 + \eta_2 \bar{\mathcal{A}}_1) y} (1 + \epsilon \lambda_m),$$

where  $\lambda_m = \frac{2a\eta_2 \xi_1 T + \frac{1}{4} \gamma \eta_2^2 \xi_1^2 (3T^2 - 4T + 1) / \hbar - \frac{1}{2} \gamma \eta_2 \xi_1^2 (3T - 1) x \bar{\mathcal{A}}_2 - \frac{1}{2} \gamma \eta_2^2 \xi_1 (3T - 1) y \bar{\mathcal{A}}_1 + \gamma \eta_2 \xi_1 x y \hbar \bar{\mathcal{A}}_1 \bar{\mathcal{A}}_2}$ . Similar formulas delight us for  ${}^i \Delta$  and  ${}^i S$ .

**A generic morphism.**

