$$\mathbb{E}\Big[a_3\alpha_1+a_3\alpha_2+t_3(\tau_1+\tau_2),\ y_3\eta_1+e^{-\gamma\alpha_1}y_3\eta_2+e^{-\gamma\alpha_2}x_3\xi_1+\frac{(1-T_3)\eta_2\xi_1}{\hbar}+x_3\xi_2,$$
$$1+\frac{1}{4\hbar}\eta_2\xi_1\big(8\hbar a_3T_3+4e^{-\gamma\alpha_1-\gamma\alpha_2}\gamma\hbar^2x_3y_3+2e^{-\gamma\alpha_1}\gamma\hbar y_3\eta_2-6e^{-\gamma\alpha_1}\gamma\hbar T_3y_3\eta_2+2e^{-\gamma\alpha_2}\gamma\hbar x_3\xi_1-6e^{-\gamma\alpha_2}\gamma\hbar T_3x_3\xi_1+\gamma\eta_2\xi_1-4\gamma T_3\eta_2\xi_1+3\gamma T_3^2\eta_2\xi_1\big)\epsilon+O[\epsilon]^2\Big]$$

```
S[U_, kk_] := S[U, kk] = Module[{OE},
   OE = m_{3,2,1→1}[Exp_{QU1,$k}[η, S1[QU[y1]] /. QU → Times]
       Exp_{QU2,$k}[α, S2[QU[a2]] /. QU → Times]
       Exp_{QU3,$k}[ξ, S3[QU[x3]] /. QU → Times]];
   E[-t1 τ1 + OE[[1]], OE[[2]], OE[[3]]] /.
    {η → η1, α → α1, ξ → ξ1}];
tS_i_ := S[$U, $k] /. {(v : τ | η | α | ξ)_1 → v_i,
    (v : t | T | y | a | x)_1 → v_i};
```

```
tS1
```

$$\mathbb{E}\Big[-a_1\alpha_1-t_1\tau_1,\ \frac{-e^{\gamma\alpha_1}\hbar y_1\eta_1-e^{\gamma\alpha_1}\hbar T_1x_1\xi_1+e^{\gamma\alpha_1}\eta_1\xi_1-e^{\gamma\alpha_1}T_1\eta_1\xi_1}{\hbar T_1},\ 1+$$
$$\frac{1}{4\hbar T_1^2}\big(4e^{\gamma\alpha_1}\gamma\hbar^2T_1y_1\eta_1-4e^{\gamma\alpha_1}\hbar^2a_1T_1y_1\eta_1-2e^{2\gamma\alpha_1}\gamma\hbar^2y_1^2\eta_1^2-4e^{\gamma\alpha_1}\hbar^2a_1T_1^2x_1\xi_1-4e^{\gamma\alpha_1}\gamma\hbar T_1\eta_1\xi_1+8e^{\gamma\alpha_1}\hbar a_1T_1\eta_1\xi_1+4e^{\gamma\alpha_1}\gamma\hbar T_1^2\eta_1\xi_1-4e^{2\gamma\alpha_1}\gamma\hbar^2T_1x_1y_1\eta_1\xi_1+6e^{2\gamma\alpha_1}\gamma\hbar y_1\eta_1^2\xi_1-2e^{2\gamma\alpha_1}\gamma\hbar T_1y_1\eta_1^2\xi_1-2e^{2\gamma\alpha_1}\gamma\hbar^2T_1^2x_1^2\xi_1^2+6e^{2\gamma\alpha_1}\gamma\hbar T_1x_1\eta_1\xi_1^2-2e^{2\gamma\alpha_1}\gamma\hbar T_1^2x_1\eta_1\xi_1^2-3e^{2\gamma\alpha_1}\gamma\eta_1^2\xi_1^2+4e^{2\gamma\alpha_1}\gamma T_1\eta_1^2\xi_1^2-e^{2\gamma\alpha_1}\gamma T_1^2\eta_1^2\xi_1^2\big)\epsilon+O[\epsilon]^2\Big]$$

```
Δ[U_, kk_] := Δ[U, kk] = Module[{OE},
   OE = Block[{$k = kk, $p = kk + 1},
     m_{1,3,5→1}@
      m_{2,4,6→2}@Times[ (* Warning:
         wrong unless $p≥$k+1! *)
        ReplacePart[1 → 0]@
         Exp_{QU1,$k}[η, Δ_{1→1,2}[QU[y1]] /. QU → Times],
        ReplacePart[2 → 0]@
         Exp_{QU3,$k}[α, Δ_{3→3,4}[QU[a3]] /. QU → Times],
        ReplacePart[1 → 0]@
         Exp_{QU5,$k}[ξ, Δ_{5→5,6}[QU[x5]] /. QU → Times]
       ] /. {η → η1, α → α1, ξ → ξ1}];
   E[τ1 (t1 + t2) + α1 (a1 + a2), OE[[2]], OE[[3]]]];
tΔ_{i_→j_,k_} :=
  Δ[$U, $k] /. {(v : τ | η | α | ξ)_1 → v_i,
    (v : t | T | y | a | x)_1 → v_j, (v : t | T | y | a | x)_2 → v_k};
```

```
tΔ_{1→1,2}
```

$$\mathbb{E}\Big[(a_1+a_2)\alpha_1+(t_1+t_2)\tau_1,\ y_1\eta_1+T_1y_2\eta_1+x_1\xi_1+x_2\xi_1,$$
$$1+\frac{1}{2}\big(-2\hbar a_1T_1y_2\eta_1+\gamma\hbar T_1y_1y_2\eta_1^2-2\hbar a_1x_2\xi_1+\gamma\hbar x_1x_2\xi_1^2\big)\epsilon+O[\epsilon]^2\Big]$$

The Faddeev-Quesne formula:

$$\mathbb{e}_{q\_,k\_}[x\_] := e^\wedge\left(\sum_{j=1}^{k+1}\frac{(1-q)^j x^j}{j\,(1-q^j)}\right);\quad \mathbb{e}_{q\_}[x\_] := \mathbb{e}_{q,\$k}[x]$$

```
R[QU, kk_] :=
  R[QU, kk] = E[-ħ a2 t1/γ, ħ x2 y1,
    Series[e^{ħ γ^-1 t1 a2 - ħ y1 x2}
      (e^{ħ b1 a2} 𝕖_{q_ħ,kk}[ħ y1 x2] /. b1 → γ^-1 (ϵ a1 - t1)),
     {ϵ, 0, kk}]];
tR_{i_,j_} :=
  R[$U, $k] /. {(v : t | T | y | a | x)_1 → v_i,
    (v : t | T | y | a | x)_2 → v_j};
t̄R_{i_,j_} := t̄R_{i,j} = tR_{i,j} ~B_j~ tS_j;
```

```
{tR_{1,2}, t̄R_{1,2}}
```

$$\Big\{\mathbb{E}\Big[-\frac{\hbar a_2t_1}{\gamma},\ \hbar x_2y_1,\ 1+\Big(\frac{\hbar a_1a_2}{\gamma}-\frac{1}{4}\gamma\hbar^3x_2^2y_1^2\Big)\epsilon+O[\epsilon]^2\Big],$$
$$\mathbb{E}\Big[\frac{\hbar a_2t_1}{\gamma},\ -\frac{\hbar x_2y_1}{T_1},\ 1+\frac{1}{4\gamma T_1^2}\big(-4\hbar a_1a_2T_1^2-4\gamma\hbar^2a_1T_1x_2y_1-4\gamma\hbar^2a_2T_1x_2y_1-3\gamma^2\hbar^3x_2^2y_1^2\big)\epsilon+O[\epsilon]^2\Big]\Big\}$$

tC is the counterclockwise spinner; $\overline{\text{tC}}$ is its inverse.

```
tC_i_ := E[0, 0, T_i^{1/2} e^{-ϵ a_i ħ} + 0_{$k}];
t̄C_i_ := E[0, 0, T_i^{-1/2} e^{ϵ a_i ħ} + 0_{$k}];
```

```
Block[{$k = 3}, {tC1, t̄C2}]
```

$$\Big\{\mathbb{E}\Big[0,0,\sqrt{T_1}-\hbar a_1\sqrt{T_1}\,\epsilon+\frac{1}{2}\hbar^2a_1^2\sqrt{T_1}\,\epsilon^2-\frac{1}{6}\big(\hbar^3a_1^3\sqrt{T_1}\big)\epsilon^3+O[\epsilon]^4\Big],$$
$$\mathbb{E}\Big[0,0,\frac{1}{\sqrt{T_2}}+\frac{\hbar a_2\epsilon}{\sqrt{T_2}}+\frac{\hbar^2a_2^2\epsilon^2}{2\sqrt{T_2}}+\frac{\hbar^3a_2^3\epsilon^3}{6\sqrt{T_2}}+O[\epsilon]^4\Big]\Big\}$$

```
Kink[QU, kk_] :=
  Kink[QU, kk] =
   Block[{$k = kk}, (tR_{1,3} t̄C2) ~B_{1,2}~ tm_{1,2→1} ~B_{1,3}~ tm_{1,3→1}];
tKink_i_ := Kink[$U, $k] /. {(v : t | T | y | a | x)_1 → v_i};
K̄ink[QU, kk_] :=
  K̄ink[QU, kk] =
   Block[{$k = kk}, (t̄R_{1,3} tC2) ~B_{1,2}~ tm_{1,2→1} ~B_{1,3}~ tm_{1,3→1}];
t̄Kink_i_ := K̄ink[$U, $k] /. {(v : t | T | y | a | x)_1 → v_i}
```

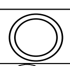
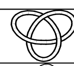
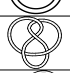
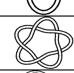
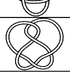
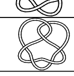
## Alternative Algorithms

```
λ_{alt,k_}[CU] := If[k == 0, 1, Module[{eq, d, b, c, so},
    eq = ρ@e^{ξ xCU}.ρ@e^{η yCU} == ρ@e^{d yCU}.ρ@e^{c (t 1CU - 2 ϵ aCU)}.ρ@e^{b xCU};
    {so} = Solve[Thread[Flatten /@ eq], {d, b, c}] /.
      C@1 → 0;
    Series[e^{-η y - ξ x + η ξ t + c t + d y - 2 ϵ c a + b x} /. so, {ϵ, 0, k}]]];
```

## The Trefoil

```
Block[{$k = 1},
 Z = tR_{1,5} tR_{6,2} tR_{3,7} t̄C4 t̄Kink8 t̄Kink9 t̄Kink10;
 Do[Z = Z ~B_{1,k}~ tm_{1,k→1}, {k, 2, 10}]; Z]
```

$$\mathbb{E}\Big[0,0,\frac{T_1}{1-T_1+T_1^2}+\big(\big(-2\hbar a_1T_1-\gamma\hbar T_1^2+2\hbar a_1T_1^2+2\gamma\hbar T_1^3-3\gamma\hbar T_1^4-2\hbar a_1T_1^4+2\gamma\hbar T_1^5+2\hbar a_1T_1^5-2\gamma\hbar^2T_1x_1y_1-2\gamma\hbar^2T_1^4x_1y_1\big)\epsilon\big)\Big/\big(1-3T_1+6T_1^2-7T_1^3+6T_1^4-3T_1^5+T_1^6\big)+O[\epsilon]^2\Big]$$

| diagram | $n_k^t$ Alexander's $\omega^+$ / Today's / Rozansky's $\rho_1^+$ | genus / ribbon / unknotting number / amphicheiral | diagram | $n_k^t$ Alexander's $\omega^+$ / Today's / Rozansky's $\rho_1^+$ | genus / ribbon / unknotting number / amphicheiral |
|---|---|---|---|---|---|
|  | $0_1^a$ 1 / 0 | 0 / ✔ / 0 / ✔ |  | $3_1^a$ $t-1$ / $t$ | 1 / ✘ / 1 / ✘ |
|  | $4_1^a$ $3-t$ / 0 | 1 / ✘ / 1 / ✔ |  | $5_1^a$ $t^2-t+1$ / $2t^3+3t$ | 2 / ✘ / 2 / ✘ |
|  | $5_2^a$ $2t-3$ / $5t-4$ | 1 / ✘ / 1 / ✘ |  | $6_1^a$ $5-2t$ / $t-4$ | 1 / ✔ / 1 / ✘ |

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Matemale-1804/