**The PBW Problem.** In $\mathcal{U}(\mathfrak{g}^\epsilon)$, bring $Z = y^3a^2x^2 \cdot y^2a^2x$ to $yax$-order. In other words, find $g \in \mathbb{Z}[\epsilon, t, y, a, x]$ such that $Z = \mathbb{O}(f = y_1^3y_2^2a_1^2a_2^2x_1^2x_2 \colon y_1a_1x_1y_2a_2x_2) = \mathbb{O}(g\colon yax)$.

**Solution, Part 1.** In $\hat{\mathcal{U}}(\mathfrak{g}^\epsilon)$ we have

$$X_{\tau_1,\eta_1,\alpha_1,\xi_1,\tau_2,\eta_2,\alpha_2,\xi_2} := e^{\tau_1 t}e^{\eta_1 y}e^{\alpha_1 a}e^{\xi_1 x}e^{\tau_2 t}e^{\eta_2 y}e^{\alpha_2 a}e^{\xi_2 x} = e^{\tau t}e^{\eta y}e^{\alpha a}e^{\xi x} =: Y_{\tau,\eta,\alpha,\xi},$$

where $\tau,\eta,\alpha,\xi$ are ugly functions of $\tau_1,\eta_i,\alpha_i,\xi_i$:

$$\tau = \tau_1+\tau_2-\frac{\log(1-\epsilon\eta_2\xi_1)}{\epsilon} = \tau_1+\tau_2+\eta_2\xi_1+\frac{\epsilon}{2}\eta_2^2\xi_1^2+\dots,$$

$$\eta = \eta_1+\frac{e^{-\alpha_1}\eta_2}{(1-\epsilon\eta_2\xi_1)} = \eta_1+e^{-\alpha_1}\eta_2+\epsilon e^{-\alpha_1}\eta_2^2\xi_1+\dots,$$

$$\alpha = \alpha_1+\alpha_2+2\log(1-\epsilon\eta_2\xi_1) = \alpha_1+\alpha_2-2\epsilon\eta_2\xi_1+\dots,$$

$$\xi = \frac{e^{-\alpha_2}\xi_1}{(1-\epsilon\eta_2\xi_1)}+\xi_2 = e^{-\alpha_2}\xi_1+\xi_2+\epsilon e^{-\alpha_2}\eta_2\xi_1^2+\dots.$$

**Note 1.** This defines a mapping $\Phi\colon \mathbb{R}^8_{\tau_1,\eta_1,\alpha_1,\xi_1,\tau_2,\eta_2,\alpha_2,\xi_2} \to \mathbb{R}^4_{\tau,\eta,\alpha,\xi}$.

**Proof.** $\mathfrak{g}^\epsilon$ has a 2D representation $\rho$:

```
ρt = (1 0; 0 1); ρy = (0 0; -ϵ 0);
ρa = ((1 + 1/ϵ)/2   0; 0   -(1 - 1/ϵ)/2); ρx = (0 1; 0 0);
Simplify@{ρa.ρx - ρx.ρa == ρx, ρa.ρy - ρy.ρa == -ρy,
  ρx.ρy - ρy.ρx == ρt - 2 ϵ ρa}

{True, True, True}
```

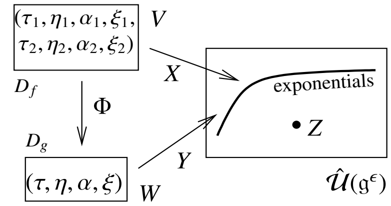
It is enough to verify the desired identity in $\rho$:

```
ME = MatrixExp;
Simplify[
 ME[τ1 ρt].ME[η1 ρy].ME[α1 ρa].ME[ξ1 ρx].ME[τ2 ρt].
   ME[η2 ρy].ME[α2 ρa].ME[ξ2 ρx] ==
  ME[τ0 ρt].ME[η0 ρy].ME[α0 ρa].ME[ξ0 ρx] /.
 {τ0 → -Log[1-ϵ η2 ξ1]/ϵ + τ1 + τ2, η0 → η1 + ℯ^-α1 η2/(1-ϵ η2 ξ1),
  α0 → 2 Log[1 - ϵ η2 ξ1] + α1 + α2, ξ0 → ℯ^-α2 ξ1/(1-ϵ η2 ξ1) + ξ2}]

True
```

**Solution, Part 2.** But now, with $D_f = f(z \mapsto \partial_\zeta) = \partial_{\eta_1}^3\partial_{\alpha_1}^2\partial_{\xi_1}^2\partial_{\eta_2}^2\partial_{\alpha_2}^2\partial_{\xi_2}$,

$$Z = D_f X_{\tau_1,\eta_1,\alpha_1,\xi_1,\tau_2,\eta_2,\alpha_2,\xi_2}\big|_{\nu s=0} = D_f Y_{\tau,\eta,\alpha,\xi}\big|_{\nu s=0} = \mathbb{O}\left(D_f e^{\tau t}e^{\eta y}e^{\alpha a}e^{\xi x}\big|_{\nu s=0} \colon yax\right) = \mathbb{O}(g\colon yax):$$

```
Expand[∂{η1,3} ∂{α1,2} ∂{ξ1,2} ∂{η2,2} ∂{α2,2} ∂{ξ2,1} Exp[
   (-Log[1-ϵ η2 ξ1]/ϵ + τ1 + τ2) t + (η1 + ℯ^-α1 η2/(1-ϵ η2 ξ1)) y +
    (2 Log[1 - ϵ η2 ξ1] + α1 + α2) a + (ℯ^-α2 ξ1/(1-ϵ η2 ξ1) + ξ2) x
  ] /. (τ | η | α | ξ)1|2 → 0]

2 a^4 t^2 x y^3 + 4 t x^2 y^4 - 16 a t x^2 y^4 + 24 a^2 t x^2 y^4 - 16 a^3 t x^2 y^4 +
 4 a^4 t x^2 y^4 + 16 x^3 y^5 - 32 a x^3 y^5 + 24 a^2 x^3 y^5 - 8 a^3 x^3 y^5 + a^4 x^3 y^5 +
 2 a^4 t x y^3 ϵ - 8 a^5 t x y^3 ϵ + 8 x^2 y^4 ϵ - 40 a x^2 y^4 ϵ + 80 a^2 x^2 y^4 ϵ -
 80 a^3 x^2 y^4 ϵ + 40 a^4 x^2 y^4 ϵ - 8 a^5 x^2 y^4 ϵ - 4 a^5 x y^3 ϵ^2 + 8 a^6 x y^3 ϵ^2
```




**Note 2.** Replacing $f \to D_f$ (and likewise $g \to D_g$), we find that $D_g = \Phi_* D_f$.

**Note 3.** The two great e-vils of mathematics are non-commutativity and non-linearity. We traded one for the other.

**Note 4.** We could have done similarly with $e^{\tau_1 t}e^{\eta_1 y}e^{\alpha_1 a}e^{\xi_1 x} = e^{\tau t+\eta y+\alpha a+\xi x}$, and with $S(e^{\tau_1 t}e^{\eta_1 y}e^{\alpha_1 a}e^{\xi_1 x})$, $\Delta(e^{\tau_1 t}e^{\eta_1 y}e^{\alpha_1 a}e^{\xi_1 x})$, $\prod_{i=1}^5 e^{\tau_i t}e^{\eta_i y}e^{\alpha_i a}e^{\xi_i x}$.

**Fact.** $R_{12} \to \exp\left(\partial_{\tau_1}\partial_{\alpha_2}+\partial_{y_1}\partial_{x_2}\right)(1+\sum_{d\geq 1}\epsilon^d p_d)$, where the $p_d$ are computable polynomials of a-priori bounded degrees.

**Moral.** We need to understand the pushforwards via maps like $\Phi$ of (formally $\infty$-order) "differential operators at 0", that in themselves are perturbed Gaussians. This turns out to be the same problem as "0-dimensional QFT" (except no integration is ever needed), and if $\epsilon^{k+1} = 0$, it is explicitly soluble.

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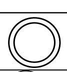
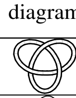
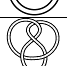
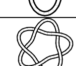
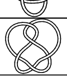
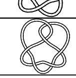
dog·ma (dôg'mə, dŏg'-) The Free Dictionary, ωεβ/TFD

*n. pl.* **dog·mas** or **dog·ma·ta** (-mə-tə)

**1.** A doctrine or a corpus of doctrines relating to matters such as morality and faith, set forth in an authoritative manner by a religion.

**2.** A principle or statement of ideas, or a group of such principles or statements, especially when considered to be authoritative or accepted uncritically: *"Much education consists in the instilling of unfounded dogmas in place of a spirit of inquiry" (Bertrand Russell).*

| diagram | $n_k^t$ Alexander's $\omega^+$ / Today's / Rozansky's $\rho_1^+$ | genus / ribbon / unknotting number / amphicheiral | diagram | $n_k^t$ Alexander's $\omega^+$ / Today's / Rozansky's $\rho_1^+$ | genus / ribbon / unknotting number / amphicheiral |
|---|---|---|---|---|---|
|  | $0_1^a$ 1 <br> 0 | 0 / ✔ <br> 0 / ✔ |  | $3_1^a$ $t-1$ <br> $t$ | 1 / ✘ <br> 1 / ✘ |
|  | $4_1^a$ $3-t$ <br> 0 | 1 / ✘ <br> 1 / ✔ |  | $5_1^a$ $t^2-t+1$ <br> $2t^3+3t$ | 2 / ✘ <br> 2 / ✘ |
|  | $5_2^a$ $2t-3$ <br> $5t-4$ | 1 / ✘ <br> 1 / ✘ |  | $6_1^a$ $5-2t$ <br> $t-4$ | 1 / ✔ <br> 1 / ✘ |

Video and more at `http://www.math.toronto.edu/~drorbn/Talks/LesDiablerets-1708/`