1-Smidgen sl_2 Let $\mathfrak{g}_1$ be the 4-dimensional Lie algebra $\mathfrak{g}_1 = \langle h, e', l, f\rangle$ over the ring $R = \mathbb{Q}[\epsilon]/(\epsilon^2 = 0)$, with h central and with $[f, l] = f$, $[e', l] = -e'$, and $[e', f] = h - 2\epsilon l$. Over $\mathbb{Q}$, $\mathfrak{g}_1$ is a solvable approximation of sl_2: $\mathfrak{g}_1 \supset \langle h, e', f, \epsilon h, \epsilon e', \epsilon l, \epsilon f\rangle \supset \langle h, \epsilon h, \epsilon e', \epsilon l, \epsilon f\rangle \supset 0$. Pragmatics: declare $\deg(h, e', l, f, \epsilon) = (1, 1, 0, 0, 1)$ and set $t := \mathbb{e}^h$ and $e := (t-1)e'/h$.

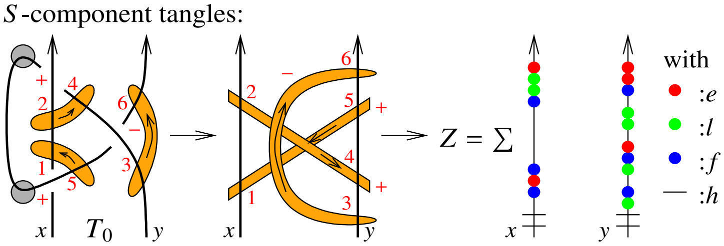
How did it arise? $sl_2 = \mathfrak{b}^+ \oplus \mathfrak{b}^-/\mathfrak{h} =: sl_2^+/\mathfrak{h}$, where $\mathfrak{b}^+ = \langle l, f\rangle/[f, l] = f$ is a Lie bialgebra with $\delta\colon \mathfrak{b}^+ \to \mathfrak{b}^+ \otimes \mathfrak{b}^+$ by $\delta\colon (l, f) \mapsto (0, l \wedge f)$. Going back, $sl_2^+ = \mathcal{D}(\mathfrak{b}^+) = (\mathfrak{b}^+)^* \oplus \mathfrak{b}^+ = \langle h', e', l, f\rangle/\cdots$. Idea. Replace $\delta \to \epsilon\delta$ over $\mathbb{Q}[\epsilon]/(\epsilon^{k+1} = 0)$. At $k = 1$, get $[f, l] = f$, $[f, h'] = -\epsilon f$, $[l, e'] = e'$, $[h', e'] = -\epsilon e'$, $[h', l] = 0$, and $[e', f] = h' - \epsilon l$. Now note that $h' + \epsilon l$ is central, so switch to $h := h' + \epsilon l$. This is $\mathfrak{g}_1$.

Ordering Symbols. $\mathbb{O}(poly \mid specs)$ plants the variables of $poly$ in $\hat{\mathcal{S}}(\oplus_i \mathfrak{g})$ along $\hat{\mathcal{U}}(\mathfrak{g})$ according to $specs$. E.g.,

$$\mathbb{O}\left(e_1 \mathbb{e}^{e_3} l_1^3 l_2 f_3^9 \mid f_3 l_1 e_1 e_3 l_2\right) = f^9 l^3 e \mathbb{e}^e l \in \hat{\mathcal{U}}(\mathfrak{g}).$$

This enables the description of elements of $\hat{\mathcal{U}}(\mathfrak{g})$ using commutative polynomials / power series. In $\mathfrak{g}_1$, no need to specify h / t.

Algebras and Invariants. Given any unital algebra A (even better if A is Hopf; typically, $A \sim \hat{\mathcal{U}}(\mathfrak{g})$), appropriate orange $R \in A \otimes A$, and appropriate cuaps $\in A$, get an $A^{\otimes S}$-valued invariant of pure S-component tangles:



What we didn't say (more, including videos, in ωεβ/Talks).

- ρ_1 is "line" in the coloured Jones polynomial; related to Melvin-Morton-Rozansky.
- ρ_1 extends to "rotational virtual tangles" and is a projection of the universal finite type invariant of such.
- ρ_1 seems to have a better chance than anything else we know to detect a counterexample to slice=ribbon.
- ρ_1 leads to many questions and a very long to-do list. Years of work, many papers ahead. Have fun!

Demo Programs. ωεβ/Demo

Formatting (prints differ ☹)

```
CF[ℰ_] := Module[{vars = Union@Cases[ℰ, e_ | l_ | f_, ∞]},
  If[vars === {}, Factor[ℰ],
   Total[CoefficientRules[ℰ, vars] /.
     (p_ → c_) ⧴ Factor[c] Times @@ (vars^p)] ]];
CF[ℰ_𝔼] := CF /@ ℰ;
```

Preparation

```
𝔼[i_, j_, s_] := 𝔼[1, (-1)^s l_j, (-t)^s e_i f_j,
   t^s e_i l_(1+s) i-s j f_j + (-1)^s l_i l_j + (-t^2)^s e_i^2 f_j^2/4];
𝔼[i_, s_] := 𝔼[1, 0, 0, s l_i];
𝔼 /: 𝔼[1, L1_, Q1_, P1_] 𝔼[1, L2_, Q2_, P2_] :=
  𝔼[1, L1 + L2, Q1 + Q2, P1 + P2];
```

Preparing the Trefoil

```
z1 = (𝔼[1, 11, 0] 𝔼[4, 2, -1] 𝔼[15, 5, 0]
   𝔼[6, 8, -1] 𝔼[9, 16, 0] 𝔼[12, 14, -1] 𝔼[3, -1] 𝔼[7, +1]
   𝔼[10, -1] 𝔼[13, +1])
```

$$\mathbb{E}\Big[1,\ -l_2 + l_5 - l_8 + l_{11} - l_{14} + l_{16},\ -\frac{e_4 f_2}{t} + e_{15} f_5 - \frac{e_6 f_8}{t} + e_1 f_{11} - \frac{e_{12} f_{14}}{t} + e_9 f_{16},\ -\frac{e_4^2 f_2^2}{4t^2} + \frac{1}{4} e_{15}^2 f_5^2 - \frac{e_6^2 f_8^2}{4t^2} + \frac{1}{4} e_1^2 f_{11}^2 - \frac{e_{12}^2 f_{14}^2}{4t^2} + \frac{1}{4} e_9^2 f_{16}^2 + e_1 f_{11} l_1 + \frac{e_4 f_2 l_2}{t} - l_3 - l_2 l_4 + l_7 + \frac{e_6 f_8 l_8}{t} - l_6 l_8 + e_9 f_{16} l_9 - l_{10} + l_1 l_{11} + l_{13} + \frac{e_{12} f_{14} l_{14}}{t} - l_{12} l_{14} + e_{15} f_5 l_{15} + l_5 l_{15} + l_9 l_{16}\Big]$$

Differential Polynomials (Implementing $P(\partial_\alpha, \partial_\beta)(f)$)

```
DP_{x_→D_α_, y_→D_β_}[P_][f_] :=
 Total[CoefficientRules[P, {x, y}] /.
   ({m_, n_} → c_) ⧴ c D[f, {α, m}, {β, n}]]
```

le and *fl* Sorts

```
S_{l_j_ (x:e|f)_i_→k_}[𝔼[ω_, L_, Q_, P_]] :=
  With[{λ = ∂_{l_j} L, α = ∂_{x_i} Q, q = 𝕖^γ β x_k + γ l_k}, CF[
    𝔼[ω, L /. l_j → l_k, t^λ α x_k + (Q /. x_i → 0),
     𝕖^-q DP_{l_j→D_γ, x_i→D_β}[P][𝕖^q] /. {β → α/ω, γ → λ Log[t]}] ]];
```

The Λόγος

```
Λ[k_] := ((t - 1) (2 (α β + δ μ)^2 - α^2 β^2) - 4 e_k l_k f_k δ^2 μ^2 -
    δ (1 + μ) (f_k^2 α^2 + e_k^2 β^2) - e_k^2 f_k^2 δ^3 (1 + 3 μ) -
    2 (α β + 2 δ μ + e_k f_k δ^2 (1 + 2 μ) + 2 l_k δ μ^2) (f_k α + e_k β) -
    4 (l_k μ^2 + e_k f_k δ (1 + μ)) (α β + δ μ)) (1 + t) / 4;
```

fe Sorts

```
S_{f_i_ e_j_→k_}[𝔼[ω_, L_, Q_, P_]] :=
  With[{q = ((1 - t) α β + β e_k + α f_k + δ e_k f_k) / μ}, CF[
    𝔼[μ ω, L, μ ω q + μ (Q /. f_i | e_j → 0),
      μ^4 𝕖^-q DP_{f_i→D_α, e_j→D_β}[P][𝕖^q] + ω^4 Λ[k]] /. μ → 1 + (t - 1) δ /.
     {α → ω^-1 (∂_{f_i} Q /. e_j → 0), β → ω^-1 (∂_{e_j} Q /. f_i → 0),
      δ → ω^-1 ∂_{f_i,e_j} Q}]];
```

Elf Merges

```
m_{i_,j_→k_}[Z_𝔼] := Module[{x, z},
  CF[(Z // S_{f_i e_j→x} // S_{l_i e_x→x} // S_{f_x l_j→x}) /. z_i|j|x → z_k]]
```

Rewriting the Trefoil (by merging 16 elves)

```
(Do[z1 = z1 // m_{1,k→1}, {k, 2, 16}]; z1)
```

$$\mathbb{E}\left[\frac{1-t+t^2}{t},\ 0,\ 0,\ \frac{(-1+t)\left(1-t+t^2\right)^2\left(1-t+2t^2\right)}{t^3} - \frac{2(1+t)\left(1-t+t^2\right)^3 e_1 f_1}{t^4} - \frac{2(-1+t)(1+t)\left(1-t+t^2\right)^3 l_1}{t^4}\right]$$

Readout

```
ρ_1[𝔼[ω_, _, _, P_]] := CF[ t ((P /. e_ | l_ | f_ → 0) - t ω^3 (∂_t ω)) / ((t - 1)^2 ω^2) ]
```

$\rho_1(3_1)$

```
ρ_1[z1] // Expand
```

$$\frac{1}{t} + t$$

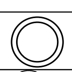
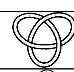
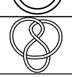
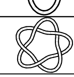
References.

[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial,* University of North Carolina PhD thesis, ωεβ/Ov.

[Ro1] L. Rozansky, *A contribution of the trivial flat connection to the Jones polynomial and Witten's invariant of 3d manifolds, I,* Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.

[Ro2] L. Rozansky, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial,* Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005.

[Ro3] L. Rozansky, *A Universal U(1)-RCC Invariant of Links and Rationality Conjecture,* arXiv:math/0201139.

diagram	n_k^t / Today's / Rozansky's ρ_1^+	Alexander's ω^+	genus / ribbon unknotting number / amphicheiral	diagram	n_k^t / Today's / Rozansky's ρ_1^+	Alexander's ω^+	genus / ribbon unknotting number / amphicheiral
	0_1^a 0	1	0 / ✔ 0 / ✔		3_1^a t	$t-1$	1 / ✘ 1 / ✘
	4_1^a 0	$3-t$	1 / ✘ 1 / ✔		5_1^a $2t^3+3t$	t^2-t+1	2 / ✘ 2 / ✘

Video and more at http://www.math.toronto.edu/~drorbn/Talks/GWU-1612/