The Algebras H and H^*. Let $q = \mathbb{e}^{\hbar\epsilon\gamma}$ and set $H = \langle a, x\rangle/([a,x] = \gamma x)$ with

$$A = \mathbb{e}^{-\hbar\epsilon a}, \quad xA = qAx, \quad S_H(a, A, x) = (-a, A^{-1}, -A^{-1}x),$$
$$\Delta_H(a, A, x) = (a_1 + a_2, A_1A_2, x_1 + A_1x_2)$$

and dual $H^* = \langle b, y\rangle/([b,y] = -\epsilon y)$ with

$$B = \mathbb{e}^{-\hbar\gamma b}, \quad By = qyB, \quad S_{H^*}(b, B, y) = (-b, B^{-1}, -yB^{-1}),$$
$$\Delta_{H^*}(b, B, y) = (b_1 + b_2, B_1B_2, y_1B_2 + y_2).$$

Pairing by $(a, x)^* = (b, y)$ ($\Rightarrow \langle B, A\rangle = q$) making $\langle y^lb^i, a^jx^k\rangle = \delta_{ij}\delta_{kl}j![k]_q!$ so $R = \sum \frac{y^kb^j\otimes a^jx^k}{j![k]_q!}$.

The Algebra QU. Using the Drinfel'd double procedure, $QU_{\gamma,\epsilon} := H^{*cop}\otimes H$ with $(\phi f)(\psi g) = \langle\psi_1 S^{-1}f_3\rangle\langle\psi_3, f_1\rangle(\phi\psi_2)(f_2g)$ and

$$S(y, b, a, x) = (-B^{-1}y, -b, -a, -A^{-1}x),$$
$$\Delta(y, b, a, x) = (y_1 + y_2B_1, b_1 + b_2, a_1 + a_2, x_1 + A_1x_2).$$

Note also that $t := \epsilon a - \gamma b$ is central and can replace b, and set $QU = QU_\epsilon = QU_{1,\epsilon}$.

The 2D Lie Algebra. One may show* that if $[a, x] = \gamma x$ then $\mathbb{e}^{\xi x}\mathbb{e}^{\alpha a} = \mathbb{e}^{\alpha a}\mathbb{e}^{\mathbb{e}^{-\gamma\alpha}\xi x}$. Ergo with

$$\begin{tikzcd} \mathbb{S}(a,x) \arrow[loop left, "SW_{ax}"] \arrow[r, shift left, "\mathbb{O}_{ax}"] \arrow[r, shift right, "\mathbb{O}_{xa}"'] & \mathcal{U}(a,x) \end{tikzcd}$$

we have $\widetilde{SW}_{ax} = \mathbb{e}^{\alpha a + \mathbb{e}^{-\gamma\alpha}\xi x}$.

$*$ Indeed $xa = (a-\gamma)x$ thus $xa^n = (a-\gamma)^nx$ thus $x\mathbb{e}^{\alpha a} = \mathbb{e}^{\alpha(a-\gamma)}x = \mathbb{e}^{-\gamma\alpha}\mathbb{e}^{\alpha a}x$ thus $x^n\mathbb{e}^{\alpha a} = \mathbb{e}^{\alpha a}(\mathbb{e}^{-\gamma\alpha})^nx^n$ thus $\mathbb{e}^{\xi x}\mathbb{e}^{\alpha a} = \mathbb{e}^{\alpha a}\mathbb{e}^{\mathbb{e}^{-\gamma\alpha}\xi x}$.

Faddeev's Formula (In as much as we can tell, first appeared without proof in Faddeev [Fa], rediscovered and proven in Quesne [Qu], and again with easier proof, in Zagier [Za]). With $[n]_q := \frac{q^n-1}{q-1}$, with $[n]_q! := [1]_q[2]_q\cdots[n]_q$ and with $\mathbb{e}_q^x := \sum_{n\geq 0}\frac{x^n}{[n]_q!}$, we have

$$\log\mathbb{e}_q^x = \sum_{k\geq 1}\frac{(1-q)^kx^k}{k(1-q^k)} = x + \frac{(1-q)^2x^2}{2(1-q^2)} + \ldots.$$

Proof. We have that $\mathbb{e}_q^x = \frac{\mathbb{e}_q^{qx}-\mathbb{e}_q^x}{qx-x}$ ("the q-derivative of $\mathbb{e}_q^x$ is itself"), and hence $\mathbb{e}_q^{qx} = (1 + (1-q)x)\mathbb{e}_q^x$, and

$$\log\mathbb{e}_q^{qx} = \log(1 + (1-q)x) + \log\mathbb{e}_q^x.$$

Writing $\log\mathbb{e}_q^x = \sum_{k\geq 1}a_kx^k$ and comparing powers of x, we get $q^ka_k = -(1-q)^k/k + a_k$, or $a_k = \frac{(1-q)^k}{k(1-q^k)}$. □

A Full Implementation. ωεβ/Full

Utilities

```
CF[sd_SeriesData] := MapAt[CF, sd, 3];
CF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[
    Expand[ℰ] //. 𝕖^x_ 𝕖^y_ :→ 𝕖^(x+y) /. 𝕖^x_ :→ 𝕖^CF[x]];
```

```
Kδ /: Kδ_(i_,j_) := If[i === j, 1, 0];
𝔼 /: 𝔼[L1_, Q1_, P1_] ≡ 𝔼[L2_, Q2_, P2_] :=
  CF[L1 == L2] ∧ CF[Q1 == Q2] ∧ CF[Normal[P1 - P2] == 0];
𝔼 /: 𝔼[L1_, Q1_, P1_] 𝔼[L2_, Q2_, P2_] :=
  𝔼[L1 + L2, Q1 + Q2, P1 * P2];
𝔼[L_, Q_, P_]_$k_ := 𝔼[L, Q, Series[Normal@P, {ϵ, 0, $k}]];
```

Zip and Bind

```
{t*, b*, y*, a*, x*, z*} = {τ, β, η, α, ξ, ζ};
{τ*, β*, η*, α*, ξ*, ζ*} = {t, b, y, a, x, z};
(u_i_)* := (u*)_i;
```

```
collect[sd_SeriesData, ζ_] :=
  MapAt[collect[#, ζ] &, sd, 3];
collect[ℰ_, ζ_] := Collect[ℰ, ζ];
Zip_{}[P_] := P; Zip_{ζ_,ζs___}[P_] :=
 (collect[P // Zip_{ζs}, ζ] /. f_. ζ^d_. :→ ∂_{ζ*,d} f) /. ζ* → 0
```

```
QZip_ζs_List@𝔼[L_, Q_, P_] :=
  Module[{ζ, z, zs, c, ys, ηs, qt, zrule, ζrule},
   zs = Table[ζ*, {ζ, ζs}];
   c = CF[Q /. Alternatives @@ (ζs ⋃ zs) → 0];
   ys = CF@Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = CF@Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
   qt = CF@Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}Q, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → CF[qt.(zs + ys)]];
   ζrule = Thread[ζs → ζs + ηs.qt];
   CF /@ 𝔼[L, c + ηs.qt.ys,
     Det[qt] Zip_ζs[P /. (zrule ⋃ ζrule)]] ];
```

```
U2l = {B_i_^p_. → 𝕖^(-p ℏ γ b_i), B^p_. → 𝕖^(-p ℏ γ b), T_i_^p_. → 𝕖^(p ℏ t_i),
   T^p_. → 𝕖^(p ℏ t), 𝒜_i_^p_. → 𝕖^(p γ α_i), 𝒜^p_. → 𝕖^(p γ α)};
l2U = {𝕖^(c_. b_i_ + d_.) :→ B_i^(-c/(ℏ γ)) 𝕖^d, 𝕖^(c_. b + d_.) :→ B^(-c/(ℏ γ)) 𝕖^d,
   𝕖^(c_. t_i_ + d_.) :→ T_i^(c/ℏ) 𝕖^d, 𝕖^(c_. t + d_.) :→ T^(c/ℏ) 𝕖^d,
   𝕖^(c_. α_i_ + d_.) :→ 𝒜_i^(c/γ) 𝕖^d, 𝕖^(c_. α + d_.) :→ 𝒜^(c/γ) 𝕖^d,
   𝕖^ℰ_ :→ 𝕖^Expand@ℰ};
```

```
LZip_ζs_List@𝔼[L_, Q_, P_] :=
  Module[{ζ, z, zs, c, ys, ηs, lt, zrule, L1, L2, Q1, Q2},
   zs = Table[ζ*, {ζ, ζs}];
   c = L /. Alternatives @@ (ζs ⋃ zs) → 0;
   ys = Table[∂_ζ (L /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂_z (L /. Alternatives @@ ζs → 0), {z, zs}];
   lt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}L, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → lt.(zs + ys)];
   L2 = (L1 = c + ηs.zs /. zrule) /. Alternatives @@ zs → 0;
   Q2 = (Q1 = Q /. U2l /. zrule) /. Alternatives @@ zs → 0;
   CF /@ 𝔼[L2, Q2, Det[lt] 𝕖^(-L2-Q2)
       Zip_ζs[𝕖^(L1+Q1) (P /. U2l /. zrule)]] //. l2U ];
```

```
B_{}[L_, R_] := L R;
B_{is__}[L_𝔼, R_𝔼] := Module[{n}, Times[
      L /. Table[(v : b | B | t | T | a | x | y)_i → v_n@i, {i, {is}}],
      R /. Table[(v : β | τ | α | 𝒜 | ξ | η)_i → v_n@i, {i, {is}}]
    ] // LZip_Join@@Table[{β_n@i, τ_n@i, a_n@i}, {i, {is}}] //
   QZip_Join@@Table[{ξ_n@i, y_n@i}, {i, {is}}] ];
B_is___[L_, R_] := B_{is}[L, R];
```

𝔼 morphisms with domain and range.

```
B_is_List[𝔼_d1_→r1_[L1_, Q1_, P1_], 𝔼_d2_→r2_[L2_, Q2_, P2_]] :=
  𝔼_(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is]) @@
   B_is[𝔼[L1, Q1, P1], 𝔼[L2, Q2, P2]];
𝔼_d1_→r1_[L1_, Q1_, P1_] // 𝔼_d2_→r2_[L2_, Q2_, P2_] :=
  B_r1⋂d2[𝔼_d1→r1[L1, Q1, P1], 𝔼_d2→r2[L2, Q2, P2]];
𝔼_d1_→r1_[L1_, Q1_, P1_] ≡ 𝔼_d2_→r2_[L2_, Q2_, P2_] ^:=
  (d1 == d2) ∧ (r1 == r2) ∧ (𝔼[L1, Q1, P1] ≡ 𝔼[L2, Q2, P2]);
𝔼_d1_→r1_[L1_, Q1_, P1_] 𝔼_d2_→r2_[L2_, Q2_, P2_] ^:=
  𝔼_(d1⋃d2)→(r1⋃r2) @@ (𝔼[L1, Q1, P1] 𝔼[L2, Q2, P2]);
𝔼_d_→r_[L_, Q_, P_]_$k_ := 𝔼_d→r @@ 𝔼[L, Q, P]_$k;
𝔼_[ℰ___][i_] := {ℰ}[[i]];
```

"Define" code

```
SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs];);
```

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Ohio-1901