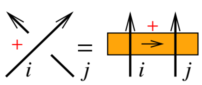
1-Smidgen sl_2 Let $\mathfrak{g}_1$ be the 4-dimensional Lie algebra $\mathfrak{g}_1 = \langle b,c,u,w\rangle$ over the ring $R=\mathbb{Q}[\epsilon]/(\epsilon^2=0)$, with b central and with $[w,c]=w$, $[c,u]=u$, and $[u,w]=b-2\epsilon c$, with CYBE $r_{ij}=(b_i-\epsilon c_i)c_j+u_iw_j$ in $\mathcal{U}(\mathfrak{g}_1)^{\otimes\{i,j\}}$. Over $\mathbb{Q}$, $\mathfrak{g}_1$ is a solvable approximation of sl_2: $\mathfrak{g}_1\supset\langle b,u,w,\epsilon b,\epsilon c,\epsilon u,\epsilon w\rangle\supset\langle b,\epsilon b,\epsilon c,\epsilon u,\epsilon w\rangle\supset 0$. (note: $\deg(b,c,u,w,\epsilon)=(1,0,1,0,1)$)

0-Smidgen sl_2 ☺. Let $\mathfrak{g}_0$ be $\mathfrak{g}_1$ at $\epsilon=0$, or $\mathbb{Q}\langle b,c,u,w\rangle/([b,\cdot]=0,\ [c,u]=u,\ [c,w]=-w,\ [u,w]=b$ with $r_{ij}=b_ic_j+u_iw_j$. It is $\mathfrak{b}^*\rtimes\mathfrak{b}$ where $\mathfrak{b}$ is the 2D Lie algebra $\mathbb{Q}\langle c,w\rangle$ and (b,u) is the dual basis of (c,w). For topology, it is more valuable than $\mathfrak{g}_1$ / sl_2, but topology already got by other means almost everything $\mathfrak{g}_0$ gives.

How did these arise? $sl_2=\mathfrak{b}^+\oplus\mathfrak{b}^-/\mathfrak{h}=:sl_2^+/\mathfrak{h}$, where $\mathfrak{b}^+=\langle c,w\rangle/[w,c]=w$ is a Lie bialgebra with $\delta\colon\mathfrak{b}^+\to\mathfrak{b}^+\otimes\mathfrak{b}^+$ by $\delta\colon(c,w)\mapsto(0,c\wedge w)$. Going back, $sl_2^+=\mathcal{D}(\mathfrak{b}^+)=(\mathfrak{b}^+)^*\oplus\mathfrak{b}^+=\langle b,u,c,w\rangle/\cdots$. **Idea.** Replace $\delta\to\epsilon\delta$ over $\mathbb{Q}[\epsilon]/(\epsilon^{k+1}=0)$. At $k=0$, get $\mathfrak{g}_0$. At $k=1$, get $[w,c]=w$, $[w,b']=-\epsilon w$, $[c,u]=u$, $[b',u]=-\epsilon u$, $[b',c]=0$, and $[u,w]=b'-\epsilon c$. Now note that $b'+\epsilon c$ is central, so switch to $b:=b'+\epsilon c$. This is $\mathfrak{g}_1$.

Ordering Symbols. $\mathbb{O}(poly\mid specs)$ plants the variables of $poly$ in $\mathcal{S}(\oplus_i\mathfrak{g})$ on several tensor copies of $\mathcal{U}(\mathfrak{g})$ according to $specs$. E.g.,
$$\mathbb{O}\left(c_1^3u_1c_2e^{u_3}w_3^9\middle|x\colon w_3c_1,\ y\colon u_1u_3c_2\right)=w^9c^3\otimes ue^uc\in\mathcal{U}(\mathfrak{g})_x\otimes\mathcal{U}(\mathfrak{g})_y$$
This enables the description of elements of $\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$ using commutative polynomials / power series.

0-Smidgen Invariants. $r=Id\in\mathfrak{b}^-\otimes\mathfrak{b}^+$ solves the CYBE $[r_{12},r_{13}]+[r_{12},r_{23}]+[r_{13},r_{23}]=0$ in $\mathcal{U}(\mathfrak{g}_0)^{\otimes 3}$ and, by luck,



$=R_{ij}=e^{r_{ij}}=e^{b_ic_j+u_iw_j}\in\mathcal{U}(\mathfrak{g}_{0,i}\oplus\mathfrak{g}_{0,j})$ solves YB/R3.

Lemma. $R_{ij}=e^{b_ic_j+u_iw_j}=\mathbb{O}\left(\exp\left(b_ic_j+\frac{e^{b_i}-1}{b_i}u_iw_j\right)\middle|i\colon u_i,\ j\colon c_jw_j\right)$

Example. $Z(T_0)=$ $\qquad=\sum_{m,n}\frac{b_i^{m-n}(e^{b_i}-1)^n}{m!n!}u^n\otimes c^mw^n$.
$$\mathbb{O}\Big(\exp\left(b_5c_1+\tfrac{e^{b_5}-1}{b_5}u_5w_1+b_2c_4+\tfrac{e^{b_2}-1}{b_2}u_2w_4-b_3c_6+\tfrac{e^{-b_3}-1}{b_3}u_3w_6\right)\Big|$$
"ucw form"
$$x\colon c_1w_1u_2,\ y\colon u_3c_4w_4u_5c_6w_6\Big)=\mathbb{O}\left(\zeta\middle|x\colon u_xc_xw_x,\ y\colon u_yc_yw_y\right)$$

Goal. Write ζ as a Gaussian: ωe^{L+Q} where L bilinear in b_i and c_i with integer coefficients, Q a balanced quadratic in u_i and w_i with coefficients in $R_S:=\mathbb{Q}(b_i,e^{b_i})$, and $\omega\in R_S$.

The Big $\mathfrak{g}_0$ Lemma. Under $[c,u]=u$, $[c,w]=-w$, and $[u,w]=b$:

1a. $N^{cu}:=\mathbb{O}(e^{\gamma c+\beta u}|uc)\stackrel{\rightarrow}{=}\mathbb{O}(e^{\gamma c+e^\gamma\beta u}|cu)$ (means $e^{\beta u}e^{\gamma c}=e^{\gamma c}e^{e^\gamma\beta u}$

1b. $N^{wc}:=\mathbb{O}(e^{\gamma c+\alpha w}|wc)\stackrel{\rightarrow}{=}\mathbb{O}(e^{\gamma c+e^\gamma\alpha w}|cw)$ …in the $\{ax+b\}$ group)

2. $\mathbb{O}(e^{\alpha w+\beta u}|wu)=\mathbb{O}(e^{-b\alpha\beta+\alpha w+\beta u}|uw)$ (the Weyl relations)

3. $\mathbb{O}(e^{\delta uw}|wu)e^{\beta u}=e^{\nu\beta u}\mathbb{O}(e^{\delta uw}|wu)$, with $\nu=(1+b\delta)^{-1}$

(a. expand and crunch. b. use $w=b\hat{x}$, $u=\partial_x$. c. use "scatter and glow".)

4. $\mathbb{O}(e^{\delta uw}|wu)=\mathbb{O}(\nu e^{\nu\delta uw}|uw)$ (same techniques)

5. $N^{wu}:=\mathbb{O}(e^{\beta u+\alpha w+\delta uw}|wu)\stackrel{\rightarrow}{=}\mathbb{O}(\nu e^{-b\nu\alpha\beta+\nu\alpha w+\nu\beta u+\nu\delta uw}|uw)$

6. $N_k^{c_ic_j}:=\mathbb{O}(\zeta|c_ic_j)\stackrel{\rightarrow}{=}\mathbb{O}(\zeta/(c_i,c_j\to c_k)|c_k)$

Sneaky. α may contain (other) u's, β may contain (other) w's.

Strand Stitching, m_k^{ij}, is defined as the composition
$$u_ic_i\overline{w_iu_j}c_jw_j\xrightarrow{N_x^{w_iu_j}}u_i\overline{c_iu_x}\,\overline{w_xc_j}w_j\xrightarrow{N_x^{c_iu_x}/\!/N_x^{w_xc_j}}\overline{u_iu_x}\,\overline{c_xc_x}\,\overline{w_xw_j}\xrightarrow{i,j,x\to k}u_kc_kw_k$$

On to 1-smidgen invariants, where much is the same…

The Big $\mathfrak{g}_1$ Lemma. Parts 1 and 6 are the same, yet

5. $\mathbb{O}\left(e^{\alpha w+\beta u+\delta uw}|wu\right)=\mathbb{O}\left(\nu(1+\epsilon\nu\Lambda)e^{\nu(-b\alpha\beta+\alpha w+\beta u+\delta uw)}|ucw\right)$

Here Λ is for Λόγος, "a principle of order and knowledge", a balanced quartic in α, β, u, c, and w:
$$\begin{aligned}\Lambda=&-b\nu(\alpha^2\beta^2\nu^2+4\alpha\beta\delta\nu+2\delta^2)/2+\beta^2\delta\nu^3(b\delta+2)u^2/2\\&+\delta^3\nu^3(3b\delta+4)u^2w^2/2+\beta\delta^2\nu^3(2b\delta+3)u^2w\\&+\alpha\delta^2\nu^3(2b\delta+3)uw^2+2\delta\nu^2(b\delta+2)(\alpha\beta\nu+\delta)uw\\&+\alpha^2\delta\nu^3(b\delta+2)w^2/2+2(\alpha\beta\nu+\delta)c+2\beta\delta\nu uc+2\delta^2\nu ucw\\&+2\alpha\delta\nu cw+\beta\nu^2(\alpha\beta\nu+2\delta)u+\alpha\nu^2(\alpha\beta\nu+2\delta)w.\end{aligned}$$

Proof. A lengthy computation. (Verification: ωεβ/Big)

Problem. We now need to normal-order perturbed Gaussians!

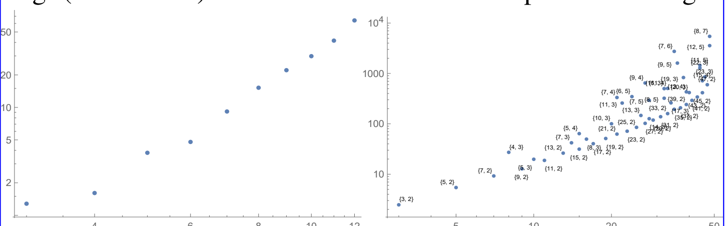
Solution. Borrow some tactics from QFT:
$$\mathbb{O}(\epsilon P(c,u)e^{\gamma c+\beta u}|uc)=\mathbb{O}(\epsilon P(\partial_\gamma,\partial_\beta)e^{\gamma c+\beta u}|uc)=$$
and likewise $\qquad\mathbb{O}(\epsilon P(\partial_\gamma,\partial_\beta)e^{\gamma c+e^{-\gamma}\beta u}|cu)$,
$$\mathbb{O}\left(\epsilon P(u,w)e^{\alpha w+\beta u+\delta uw}|wu\right)=\mathbb{O}\left(\epsilon P(\partial_\beta,\partial_\alpha)\nu e^{\nu(-b\alpha\beta+\alpha w+\beta u+\delta uw)}|ucw\right)$$

Finally, the values of the generators ⤡, ⤢, $\overrightarrow{n}$, and $\underset{\rightarrow}{u}$, are set by solving many equations, non-uniquely.

Pragmatic Simplifications. Set $t:=e^b$, work with $v:=(t-1)u/b$, and set $\mathbb{E}(\omega,L,Q,P):=\mathbb{O}\left(\omega^{-1}e^{L+Q/\omega}(1+\epsilon\omega^{-4}P)\colon(i\colon v_ic_iw_i)\right)$. Now $\omega\in R_S:=\mathbb{Z}[t_i,t_i^{-1}]$ is Laurent, $L=\sum l_{ij}\log(t_i)c_j$ with $l_{ij}\in\mathbb{Z}$, $Q=\sum q_{ij}v_iw_j$ with $q_{ij}\in R_S$, and P is a quartic polynomial in v_i, c_j, w_k with coefficients in R_S. The operations are lightly modified, and the Λόγος and the values of the generators become somewhat simpler, as in the implementation below.

Rough complexity estimate, after $t_k\to t$. n: xing number; w: width, maybe $\sim\sqrt{n}$.
$$\underbrace{n}_{A}\overbrace{\sum_{d=0}^{4}\underbrace{w^{4-d}}_{E}\underbrace{w^d}_{F}\underbrace{n^2}_{G}}^{B}=n^3w^4\in[n^5,n^7]$$
A: go over stitchings in order. B: multiplication ops per $N^{u_iw_j}$. d: deg of u_i,w_j in P. E: #terms of deg d in P. F: ops per term. G: cost per polynomial multiplication op.

Experimental Analysis (ωεβ/Exp). Log-log plots of computation time (sec) vs. crossing number, for all knots with up to 12 crossings (mean times) and for all torus knots with up to 48 crossings:



Conjecture (checked on the same collections). Given a knot K with Alexander polynomial A, there is a polynomial ρ_1 such that
$$P=A^2\frac{(t-1)^3\rho_1+t^2(2vw+(1-t)(1-2c))AA'}{(1-t)t}.$$
Furthermore, A and ρ_1 are symmetric under $t\to t^{-1}$, so let A^+ and ρ_1^+ be their "positive parts", so e.g., $\rho_1(t)=\rho_1^+(t)+\rho_1^+(t^{-1})-\rho_1^+(0)$.

Power. On the 250 knots with at most 10 crossings, the pair (A,ρ_1) attains 250 distinct values, while (Khovanov, HOMFLY-PT) attains only 249 distinct values. To 11 crossings the numbers are (802, 788, 772) and to 12 they are (2978, 2883, 2786).

Genus. Up to 12 xings, always $\deg\rho_1^+\le 2g-1$, where g is the 3-genus of K (equallity for 2530 knots). This gives a lower bound on g in terms of ρ_1 (conjectural, but undoubtedly true). This bound is often weaker than the Alexander bound, yet for 10 of the 12-xing Alexander failures it does give the right answer.

This is http://www.math.toronto.edu/~drorbn/Talks/MIT-1612/. Better videos at …/Indiana-1611/, …/LesDiablerets-1608/