**The Taylor Remainder Formulas.** Let $f$ be a smooth function, let $P_{n,a}(x)$ be the $n$th order Taylor polynomial of $f$ around $a$ and evaluated at $x$, so with $a_k = f^{(k)}(a)/k!$,

Brook Taylor

$$P_{n,a}(x) := \sum_{k=0}^{n} a_k (x-a)^k,$$

and let $R_{n,a}(x) := f(x) - P_{n,a}(x)$ be the "mistake" or "remainder term". Then

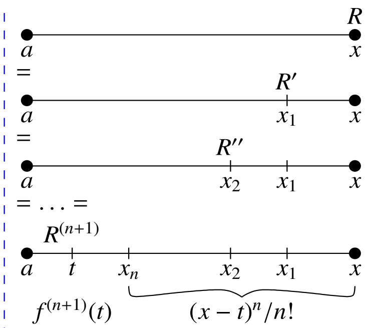
$$R_{n,a}(x) = \int_a^x dt\, \frac{f^{(n+1)}(t)}{n!}(x-t)^n, \tag{1}$$

or alternatively, for some $t$ between $a$ and $x$,

$$R_{n,a}(x) = \frac{f^{(n+1)}(t)}{(n+1)!}(x-a)^{n+1}. \tag{2}$$

(In particular, the Taylor expansions of sin, cos, exp, and of several other lovely functions converges to these functions *everywhere*, no matter the odds.)

**Proof of (1)** (for adults; I learned it from my son Itai). The fundamental theorem of calculus says that if $g(a) = 0$ then $g(x) = \int_a^x dx_1 g(x_1)$. By design, $R^{(k)}_{n,a}(a) = 0$ for $0 \le k \le n$. Therefore




$$R_{n,a}(x) = \int_a^x dx_1 R'_{n,a}(x_1)$$
$$= \int_a^x dx_1 \int_a^{x_1} dx_2 R''_{n,a}(x_2)$$
$$= \ldots = \int_a^x dx_1 \int_a^{x_1} dx_2 \ldots \int_a^{x_n} dx_n \int_a^t dt\, R^{(n+1)}_{n,a}(t)$$
$$= \int_a^x dx_1 \int_a^{x_1} dx_2 \ldots \int_a^{x_n} dx_n \int_a^t dt\, f^{(n+1)}(t),$$

when $x > a$, and with similar logic when $x < a$,

$$= \int_{a\le t\le x_n\le\ldots\le x_1\le x} f^{(n+1)}(t) = \int_a^t dt\, f^{(n+1)}(t) \int_{t\le x_n\le\ldots\le x_1\le x} 1$$
$$= \int_a^t dt \frac{f^{(n+1)}(t)}{n!} \int_{(x_1,\ldots,x_n)\in[t,x]^n} 1 = \int_a^x dt\, \frac{f^{(n+1)}(t)}{n!}(x-t)^n.$$

□

**de-Fubini** (obfuscation in the name of simplicity). Prematurely aborting the above chain of equalities, we find that for any $1 \le k \le n+1$,

Guido Fubini

$$R(x) = \int_a^x dt\, R^{(k)}(t) \frac{(x-t)^{k-1}}{(k-1)!}.$$

But these are easy to prove by induction using integration by parts, and there's no need to invoke Fubini.

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**Partial Derivatives Commute.** Make Fubini Smile Again!

If $f\colon \mathbb{R}^2 \to \mathbb{R}$ is $C^2$ near $a \in \mathbb{R}^2$, then $f_{12}(a) = f_{21}(a)$.

Proof. Let $x \in \mathbb{R}^2$ be small, and let $R := [a_1, a_1+x_1]\times[a_2, a_2+x_2]$.

$$f_{12}(a) \sim \boxed{\int f_{12}} = \sum f = \boxed{\int f_{21}} \sim f_{21}(a)$$

$$f_{12}(a) \sim \frac{1}{|R|}\int_R f_{12} = \frac{1}{|R|}\int_{a_1}^{a_1+x_1} dt_1\, (f_1(t_1, a_2+x_2) - f_1(t_1, a_2))$$
$$= \frac{1}{|R|}\begin{pmatrix} f(a_1+x_1, a_2+x_2) - f(a_1+x_1, a_2) \\ -f(a_1, a_2+x_2) + f(a_1, a_2)\end{pmatrix}.$$

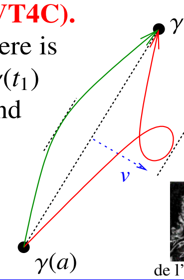
But the answer here is the same as in

$$f_{21}(a) \sim \frac{1}{|R|}\int_R f_{21} = \frac{1}{|R|}\int_{a_2}^{a_2+x_2} dt_2\, (f_2(a_1+x_1, t_2) - f_2(a_1, t_2))$$
$$= \frac{1}{|R|}\begin{pmatrix} f(a_1+x_1, a_2+x_2) - f(a_1, a_2+x_2) \\ -f(a_1+x_1, a_2) + f(a_1, a_2)\end{pmatrix},$$

and both of these approximations get better and better as $x \to 0$. □

**The Mean Value Theorem for Curves (MVT4C).** If $\gamma\colon [a,b] \to \mathbb{R}^2$ is a smooth curve, then there is some $t_1 \in (a,b)$ for which $\gamma(b) - \gamma(a)$ and $\dot\gamma(t_1)$ are linearly dependent. If also $\gamma(a) = 0$, and $\gamma = \binom{\xi}{\eta}$ and $\eta \ne 0 \ne \dot\eta$ on $(a,b)$, then

$$\frac{\xi(b)}{\eta(b)} = \frac{\dot\xi(t_1)}{\dot\eta(t_1)} \quad \left(\text{when lucky, } = \frac{\ddot\xi(t_2)}{\ddot\eta(t_2)} \ldots\right).$$




de l'Hôpital

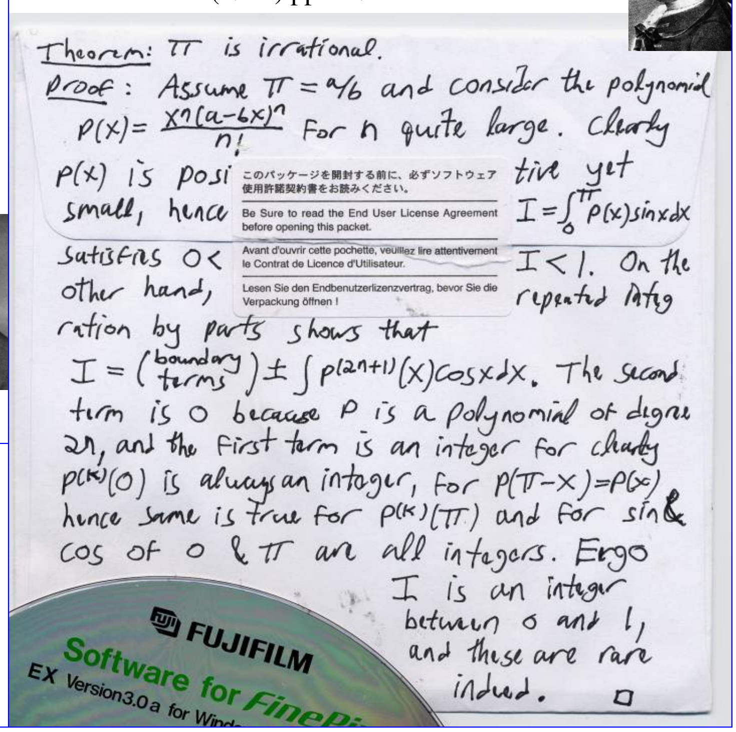
**Proof of (2).** Iterate the lucky MVT4C as follows:

$$\frac{R_{n,a}(x)}{(x-a)^{n+1}} = \frac{R'_{n,a}(t_1)}{(n+1)(t_1-a)^n} = \ldots = \frac{R^{(n+1)}_{n,a}(t_{n+1})}{(n+1)!} = \frac{f^{(n+1)}(t)}{(n+1)!}.$$

□

J.H. Lambert

**$\pi$ is Irrational** following Ivan Niven, Bull. Amer. Math. Soc. (1947) pp. 509:



Theorem: $\pi$ is irrational.

Proof: Assume $\pi = a/b$ and consider the polynomial $p(x) = \frac{x^n(a-bx)^n}{n!}$ for $n$ quite large. Clearly $p(x)$ is positive yet small, hence $I = \int_0^\pi p(x)\sin x\,dx$ satisfies $0 < I < 1$. On the other hand, repeated integration by parts shows that $I = \binom{\text{boundary}}{\text{terms}} \pm \int p^{(2n+1)}(x)\cos x\,dx$. The second term is 0 because $p$ is a polynomial of degree $2n$, and the first term is an integer for clearly $p^{(k)}(0)$ is always an integer, for $p(\pi-x)=p(x)$ hence same is true for $p^{(k)}(\pi)$ and for sin & cos of 0 & $\pi$ are all integers. Ergo $I$ is an integer between 0 and 1, and these are rare indeed. □



Video and more at http://www.math.toronto.edu/~drorbn/Talks/MAASeaway-1810/