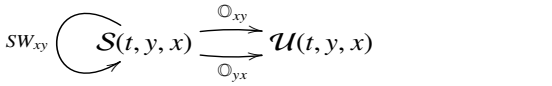
2. The "$z_i \to z_j$ variable rename map $\sigma^i_j\colon \mathcal{S}(z_i) \to \mathcal{S}(z_j)$ becomes ${}^t\sigma^i_j = \mathbb{e}^{z_j\zeta_i}$, and it's easy to rename several variables simultaneously.
3. The "archetypal multiplication map $m^{ij}_k\colon \mathcal{S}(z_i, z_j) \to \mathcal{S}(z_k)$" has ${}^t m = \mathbb{e}^{z_k(\zeta_i+\zeta_j)}$.
4. The "archetypal coproduct $\Delta^i_{jk}\colon \mathcal{S}(z_i) \to \mathcal{S}(z_j, z_k)$", given by $z_i \to z_j + z_k$ or $\Delta z = z\otimes 1 + 1\otimes z$, has ${}^t\Delta = \mathbb{e}^{(z_j+z_k)\zeta_i}$.
5. R-matrices tend to have terms of the form $\mathbb{e}_q^{\hbar y_1 x_2} \in \mathcal{U}_q \otimes \mathcal{U}_q$. The "baby R-matrix" is ${}^t R = \mathbb{e}^{\hbar yx} \in \mathcal{S}(y, x)$.

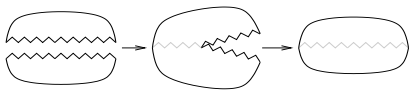
Proposition. If $F\colon \mathcal{S}(B) \to \mathcal{S}(B')$ is linear and "continuous", then ${}^t F = \exp\left(\sum_{z_i\in B} \zeta_i z_i\right) /\!\!/ F$.

The Heisenberg Example. The "Weyl form of the canonical commutation relations" states that if $[y, x] = t$ and t is central, then $\mathbb{e}^{\xi x}\mathbb{e}^{\eta y} = \mathbb{e}^{\eta y}\mathbb{e}^{\xi x}\mathbb{e}^{-\eta\xi t}$. Thus with



we have ${}^t SW_{xy} = \mathbb{e}^{\tau t+\eta y+\xi x-\eta\xi t}$.

The Zipping Issue (between unbound and bound lies half-zipped).



Zipping. If $P(\zeta^j, z_i)$ is a polynomial, or whenever otherwise convergent, set

$$\left\langle P(\zeta^j, z_i)\right\rangle_{(\zeta^j)} = P\left(\partial_{z_j}, z_i\right)\Big|_{z_i=0}.$$

(E.g., if $P = \sum a_{nm}\zeta^n z^m$ then $\langle P\rangle_\zeta = \sum n! a_{nn}$).

The Zipping / Contraction Theorem. If P has a finite ζ-degree and the y's and the q's are "small" then

$$\left\langle P(z_i, \zeta^j)\mathbb{e}^{\eta^i z_i + y_j\zeta^j}\right\rangle_{(\zeta^j)} = \left\langle P(z_i + y_i, \zeta^j)\mathbb{e}^{\eta^i(z_i+y_i)}\right\rangle_{(\zeta^j)},$$

(proof: replace $y_j \to \hbar y_j$ and test at $\hbar = 0$ and at $\partial_\hbar$), and

$$\left\langle P(z_i, \zeta^j)\mathbb{e}^{c+\eta^i z_i + y_j\zeta^j + q^i_j z_i\zeta^j}\right\rangle_{(\zeta^j)} = \det(\tilde q)\left\langle P(\tilde q^k_i(z_k + y_k), \zeta^j)\mathbb{e}^{c+\eta^i\tilde q^k_i(z_k+y_k)}\right\rangle_{(\zeta^j)}$$

where $\tilde q$ is the inverse matrix of $1 - q$: $(\delta^i_j - q^i_j)\tilde q^j_k = \delta^i_k$ (proof: replace $q^i_j \to \hbar q^i_j$ and test at $\hbar = 0$ and at $\partial_\hbar$).

Implementation. ωεβ/ZipBindDemo

```
Kδ /: Kδ_i_,j_ := If[i === j, 1, 0];
{z*, x*, y*} = {ξ, ξ, η}; {ξ*, ξ*, η*} = {z, x, y};
(u_i_)* := (u*)_i;
Zip_{} [P_] := P;
Zip_{ζ_,ζs___} [P_] :=
 (Expand[P // Zip_{ζs}] /. f_. ζ^d_. :> ∂_{ζ*,d} f) /. ζ* → 0
Zip_{ξ} [(a ξ^6 + ξ + 3) (z^5 ℮^z + 7 z) + 99 b]
```

```
7 + 720 a + 99 b
```

```
Zip_{ξ,η} [ξ^3 η^3 ℮^(a x + b y + c x y)]
```

```
a^3 b^3 + 9 a^2 b^2 c + 18 a b c^2 + 6 c^3
```

```
(* 𝔼[Q,P] means ℮^Q P *)
𝔼 /: Zip_ζs_List@𝔼[Q_, P_] :=
  Module[{ξ, z, zs, c, ys, ηs, qt, zrule, Q1, Q2},
   zs = Table[ξ*, {ξ, ζs}];
   c = Q /. Alternatives @@ (ζs ⋃ zs) → 0;
   ys = Table[∂_ξ (Q /. Alternatives @@ zs → 0), {ξ, ζs}];
   ηs = Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
   qt = Inverse@Table[Kδ_{z,ξ*} - ∂_{z,ξ} Q, {ξ, ζs}, {z, zs}];
   zrule = Thread[zs → qt.(zs + ys)];
   Q1 = c + ηs.zs /. zrule;
   Q2 = Q1 /. Alternatives @@ zs → 0;
   Simplify /@ 𝔼[Q2, Det[qt] ℮^-Q2 Zip_ζs[℮^Q1 (P /. zrule)]] ];
```

```
Eh = 𝔼[h Σ_{i=1}^{3} Σ_{j=1}^{3} a_{10 i+j} x_i ξ_j, Σ_{i=1}^{3} f_i[x_1, x_2, x_3] ξ_i];
```

```
E1 = Eh /. h → 1
```

```
𝔼[a_11 x_1 ξ_1 + a_21 x_2 ξ_1 + a_31 x_3 ξ_1 + a_12 x_1 ξ_2 +
   a_22 x_2 ξ_2 + a_32 x_3 ξ_2 + a_13 x_1 ξ_3 + a_23 x_2 ξ_3 + a_33 x_3 ξ_3,
  ξ_1 f_1[x_1, x_2, x_3] + ξ_2 f_2[x_1, x_2, x_3] + ξ_3 f_3[x_1, x_2, x_3]]
```

```
Short[lhs = Zip_{ξ_1,ξ_2}@E1, 5]
```

```
𝔼[((a_13 ((-1 + a_22) a_31 - a_21 a_32) + a_12 (-a_23 a_31 + a_21 a_33) +
        (-1 + a_11) (a_23 a_32 - (-1 + a_22) a_33)) x_3 ξ_3) /
   (-1 + a_12 a_21 - a_11 (-1 + a_22) + a_22),
                <<17>> + a_21 <<1>>
  ----------------------------------------]
                (-1 + a_12 a_21 - a_11 (-1 + a_22) + a_22)^2
```

```
lhs == Zip_{ξ_1}@Zip_{ξ_2}@E1 == Zip_{ξ_2}@Zip_{ξ_1}@E1
```

```
True
```

```
Short[
 lhs = Normal[Eh /. 𝔼[Q_, P_] :> Series[P ℮^Q, {h, 0, 3}]] //
   Zip_{ξ_1,ξ_2}, 5]
```

```
h a_13 ξ_3 f_1[0, 0, x_3] + 2 h^2 a_11 a_13 ξ_3 f_1[0, 0, x_3] +
 3 h^3 a_11^2 a_13 ξ_3 f_1[0, 0, x_3] + 2 h^3 a_12 a_13 a_21 ξ_3 f_1[0, 0, x_3] +
 h^2 a_13 a_22 ξ_3 f_1[0, 0, x_3] + <<337>> +
 1/6 h^3 a_31^3 x_3^3 ξ_3 f_3^(3,0,0)[0, 0, x_3] + 1/2 h^3 a_31^2 a_32 x_3^3 f_1^(3,1,0)[0, 0, x_3] +
 1/6 h^3 a_31^3 x_3^3 f_2^(3,1,0)[0, 0, x_3] + 1/6 h^3 a_31^3 x_3^3 f_1^(4,0,0)[0, 0, x_3]
```

```
rhs =
  Normal[Zip_{ξ_1,ξ_2}@Eh /. 𝔼[Q_, P_] :> Series[P ℮^Q, {h, 0, 3}]];
Simplify[lhs == rhs]
```

```
True
```

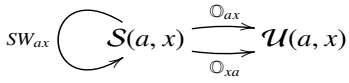
```
𝔼 /: 𝔼[Q1_, P1_] 𝔼[Q2_, P2_] := 𝔼[Q1 + Q2, P1 * P2];
Bind_{ζs_List}[L_𝔼, R_𝔼] := Module[{n, hideζs, hidezs},
   hideζs = Table[ζs[[i]] → ξ_{n@i}, {i, Length@ζs}];
   hidezs = Table[ζs[[i]]* → z_{n@i}, {i, Length@ζs}];
   Zip_{ζs}/.hideζs[(L /. hidezs) (R /. hideζs)] ];
Bind_{ξ_2}[𝔼[ξ (x_1 + x_2), 1], 𝔼[ξ_2 (x_2 + x_3), 1]]
```

```
𝔼[ξ (x_1 + x_2 + x_3), 1]
```

```
Bind_{ξ_2}[𝔼[(ξ_2 + ξ_3) x_2, 1], 𝔼[(ξ_1 + ξ_2) x, 1]]
```

```
𝔼[x (ξ_1 + ξ_2 + ξ_3), 1]
```

The 2D Lie Algebra. Clever people know* that if $[a, x] = \gamma x$ then $\mathbb{e}^{\xi x}\mathbb{e}^{\alpha a} = \mathbb{e}^{\alpha a}\mathbb{e}^{\mathbb{e}^{-\gamma\alpha}\xi x}$. Ergo with



we have ${}^t SW_{ax} = \mathbb{e}^{\alpha a + \mathbb{e}^{-\gamma\alpha}\xi x}$.

* Indeed $xa = (a - \gamma)x$ thus $xa^n = (a - \gamma)^n x$ thus $x\mathbb{e}^{\alpha a} = \mathbb{e}^{\alpha(a-\gamma)}x = \mathbb{e}^{-\gamma\alpha}\mathbb{e}^{\alpha a}x$ thus $x^n\mathbb{e}^{\alpha a} = \mathbb{e}^{\alpha a}(\mathbb{e}^{-\gamma\alpha})^n x^n$ thus $\mathbb{e}^{\xi x}\mathbb{e}^{\alpha a} = \mathbb{e}^{\alpha a}\mathbb{e}^{\mathbb{e}^{-\gamma\alpha}\xi x}$.

The Real Thing. In $QU/(\epsilon^2 = 0)$ over $\mathbb{Q}[\![\hbar]\!]$ using the *yax* order, $T = \mathbb{e}^{\hbar t}$, $\bar T = T^{-1}$, $\mathcal{A} = \mathbb{e}^{\gamma\alpha}$, and $\bar{\mathcal{A}} = \mathcal{A}^{-1}$, we have

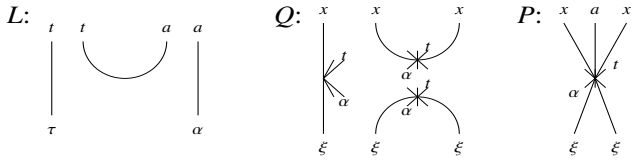
$${}^t R_{ij} = \mathbb{e}^{\hbar(y_i x_j - t_i a_j/\gamma)}\left(1 + \epsilon\hbar\left(a_i a_j/\gamma - \gamma\hbar^2 y_i^2 x_j^2/4\right)\right)$$

in $\mathcal{S}(B_i, B_j)$, and in $\mathcal{S}(B_1^*, B_2^*, B)$ we have

$${}^t m = \mathbb{e}^{(\alpha_1+\alpha_2)a+\eta_2\xi_1(1-T)/\hbar+(\xi_1\bar{\mathcal{A}}_2+\xi_2)x+(\eta_1+\eta_2\bar{\mathcal{A}}_1)y}\,(1+\epsilon\lambda_m),$$

where $\lambda_m = 2a\eta_2\xi_1 T + \frac{1}{4}\gamma\eta_2^2\xi_1^2\left(3T^2 - 4T + 1\right)/\hbar - \frac{1}{2}\gamma\eta_2\xi_1^2(3T-1)x\bar{\mathcal{A}}_2 - \frac{1}{2}\gamma\eta_2^2\xi_1(3T-1)y\bar{\mathcal{A}}_1 + \gamma\eta_2\xi_1 xy\hbar\bar{\mathcal{A}}_1\bar{\mathcal{A}}_2$. Similar formulas delight us for ${}^t\Delta$ and ${}^t S$.

A generic morphism.



Video and more at http://www.math.toronto.edu/~drorbn/Talks/Matemale-1804/