

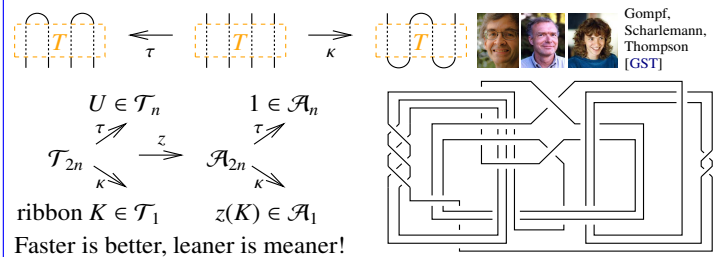
Work in Progress!

## Gauss-Gassner Invariants, What?

**Abstract.** In a “degree  $d$  Gauss diagram formula” one produces a number by summing over all possibilities of paying very close attention to  $d$  crossings in some  $n$ -crossing knot diagram while observing the rest of the diagram only very loosely, minding only its skeleton. The result is always poly-time computable as only  $\binom{n}{d}$  states need to be considered. An under-explained paper by Goussarov, Polyak, and Viro [GPV] shows that every type  $d$  knot invariant has a formula of this kind. Yet only finitely many integer invariants can be computed in this manner within any specific polynomial time bound.

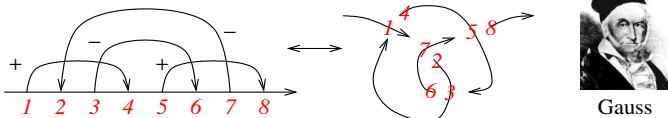
I suggest to do the same as [GPV], except replacing “the skeleton” with “the Gassner invariant”, which is still poly-time. One poly-time invariant that arises in this way is the Alexander polynomial (in itself it is infinitely many numerical invariants) and I believe (and have evidence to support my belief) that there are more.

**The QUILT Target.** QUick Invariants of Large Tangles, for little had been found since Alexander (and if they're there, how can we not know all about them?), and for {ribbon}  $\neq$  {slice}:



### Gauss Diagrams.

(just QUILK, today)



**Gauss Diagram Formulas** [PV, GPV]. If  $g$  is a Gauss diagram and  $F$  an unsigned Gauss diagram,  $\langle F, g \rangle_{PV} := \sum_{y \subseteq g} (-1)^{|y|} \delta(F, \bar{y})$ :



**Under-Explained Theorem** [GPV]. Every finite type invariant arises in this way.

$$F_2 = \text{diagram} \Rightarrow \langle F_2, K \rangle = v_2(K)$$

$$F_3 = 3 \text{diagram} + 2 \text{diagram} + \text{rotations} \Rightarrow \langle F_3, K \rangle = 6v_3(K)$$

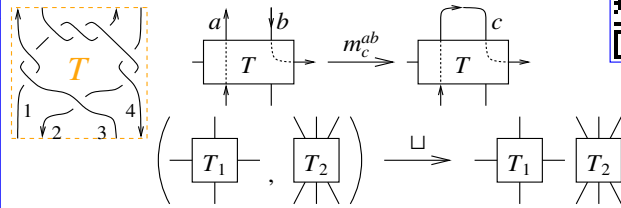
### Gauss-Gassner Invariants.

Want more? Increase your environmental awareness! Instead of nearly-forgetting  $y^c$ , compute its Burau/Gassner invariant (note that  $y^c$  is a tangle in a Swiss cheese; more easily, a virtual tangle):

$$GG_{k,F}(g) = \sum_{y \subseteq g, |y| \leq k} \bar{F}(y, z(y^c)) = \sum_{y \subseteq g, |y| \leq k} F(y, z(g \text{ cut near } y)),$$

where  $k$  is fixed and  $F(y, \gamma)$  is a function of a list of arrows  $y$  and a square matrix  $\gamma$  of side  $|y| + 1 \leq k + 1$ .

### The (Burau-)Gassner Invariant.



**Theorem 1.**  $\exists!$  an invariant  $z: \{\text{pure framed } S\text{-component tangles}\} \rightarrow \Gamma(S) := M_{S \times S}(R_S)$ , where  $R_S = \mathbb{Z}\langle (T_a)_{a \in S} \rangle$  is the ring of rational functions in  $S$  variables, intertwining

$$\left( \begin{array}{c|c} S_1 & S_2 \\ \hline S_1 & A_1 \end{array}, \begin{array}{c|c} S_2 & A_2 \\ \hline S_2 & A_2 \end{array} \right) \xrightarrow{\sqcup} \begin{array}{c|c} S_1 & S_2 \\ \hline S_1 & A_1 \quad 0 \\ S_2 & 0 \quad A_2 \end{array}$$

$$\begin{array}{c|c} a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow{m_c^{ab}} \begin{array}{c|c} c & S \\ \hline c & \gamma + \alpha\delta/\mu & \epsilon + \delta\theta/\mu \\ S & \phi + \alpha\psi/\mu & \Xi + \psi\theta/\mu \end{array}$$

and satisfying  $(|a; a \nearrow b, b \nearrow a) \xrightarrow{z} \left( \begin{array}{c|c} a & b \\ \hline a & 1 & 1 - T_a^{-1} \\ b & 0 & T_a^{-1} \end{array} \right)$

See also [LD, K LW, CT, BNS].

**Theorem 2.** With  $k = 1$  and  $F_A$  defined by

$$F_A(\xrightarrow{s}, \gamma) = s \frac{\gamma_{22}\gamma_{33} - \gamma_{23}\gamma_{32}}{\gamma_{33} + \gamma_{13}\gamma_{32} - \gamma_{12}\gamma_{33}} \Big|_{T_a \rightarrow T}$$

$$F_A(\xleftarrow{s}, \gamma) = s \frac{\gamma_{13}\gamma_{32} - \gamma_{12}\gamma_{33}}{\gamma_{32} - \gamma_{23}\gamma_{32} + \gamma_{22}\gamma_{33}} \Big|_{T_a \rightarrow T}$$

$GG_{1,F_A}(K)$  is a regular isotopy invariant. Unfortunately, for every knot  $K$ ,  $GG_{1,F_A}(K) - T \frac{d}{dT} \log A(K)(T) \in \mathbb{Z}$ , where  $A(K)$  is the Alexander polynomial of  $K$ .

**Expectation.** Higher Gauss-Gassner invariants exist ... (though right now I can reach for them only wearing my exoskeleton)

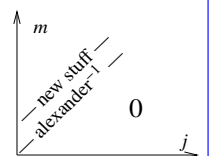


... and they are the “higher diagonals” in the MMR expansion of the coloured Jones polynomial  $J_\lambda$ .

**Theorem** ([BNG], conjectured [MM], elucidated [Ro]). Let  $J_d(K)$  be the coloured Jones polynomial of  $K$ , in the  $d$ -dimensional representation of  $sl(2)$ . Writing

$$\frac{(q^{1/2} - q^{-1/2})J_d(K)}{q^{d/2} - q^{-d/2}} \Big|_{q=e^h} = \sum_{j,m \geq 0} a_{jm}(K) d^j \hbar^m,$$

“below diagonal” coefficients vanish,  $a_{jm}(K) = 0$  if  $j > m$ , and “on diagonal” coefficients give the inverse of the Alexander polynomial:  $(\sum_{m=0}^{\infty} a_{mm}(K) \hbar^m) \cdot A(K)(e^h) = 1$ .



# Help Needed!

I'm slow and feeble-minded.