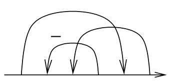
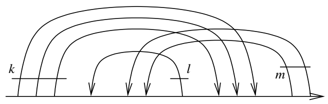
Warning. Conventions on this page change randomly from line to line.

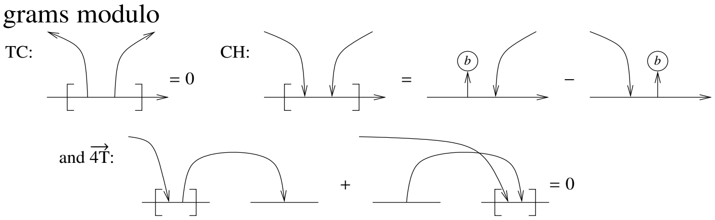
$Z^{w/2}$. The GGA story is about $Z^{w/2}\colon \mathcal{K} \to \mathcal{A}^{w/2}$, defined on arrows a by $\pm a \mapsto \exp(\pm a)$:



$$\to \sum_{k,l,m\geq 0} \frac{(+)^k(-)^l(+)^m}{k!\,l!\,m!}$$



Where the target space $\mathcal{A}^{w/2}$ is the space of unsigned arrow diagrams modulo

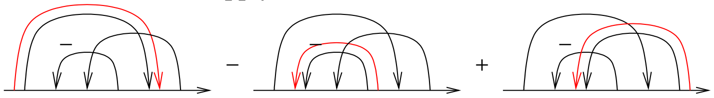


($Z^{w/2}$ is a reduction of the much-studied Z^w [BND, BN]).

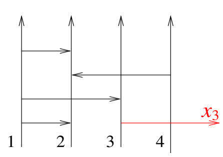
The Euler Trick. How best do non-commutative algebra with exponentials? Logarithms are from hell as $e^fe^g = e^{\mathrm{bch}(f,g)}$, but Euler's from heaven: Let E be the derivation $Ef := (\deg f)f$ $(= xf'$, in $\mathbb{Q}[\![x]\!]$) and let $\tilde{E}Z := Z^{-1}EZ$ $(= x(\log Z)'$ in same). If $\deg x = 1$ then $\tilde{E}e^x = x$ and if $F = e^f$ and $G = e^g$, then $\tilde{E}(FG)$ is

$$(FG)^{-1}((EF)G + F(EG)) = G^{-1}(\tilde{E}F)G + \tilde{E}G = e^{-\operatorname{ad} g}(\tilde{E}F) + \tilde{E}G.$$

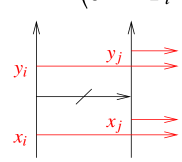
Scatter and Glow. Apply $\tilde{E}$ to $Z(K)$. EZ is shown:



Tail scattering. The algebra $\mathbb{Q}[\![b_i]\!]\langle a_{ij}\rangle$ modulo $[a_{ij}, a_{kl}] = 0$ (loc), $[a_{ij}, a_{ik}] = 0$ (TC), and $[a_{ik}, a_{jk}] = -[a_{ij}, a_{jk}] = b_ja_{ik} - b_ia_{jk}$ (CH and $\overrightarrow{4T}$), acts on $V = \mathbb{Q}[\![b_i]\!]\langle x_i = a_{i\infty}\rangle$ by $[a_{ij}, x_i] = 0$, $[a_{ij}, x_j] = b_ix_j - b_jx_i$. Hence $e^{\operatorname{ad} a_{ij}}x_i = x_i$, $e^{\operatorname{ad} a_{ij}}x_j = e^{b_i}x_j + \frac{b_j}{b_i}(1 - e^{b_i})x_i$. Renaming $\bar{x}_i = x_i/b_i$, $T_i = e^{b_i}$, get $[e^{\operatorname{ad} a_{ij}}]_{\bar{x}_i,\bar{x}_j} = \begin{pmatrix} 1 & 1-T_i \\ 0 & T_i \end{pmatrix}$. Alternatively,

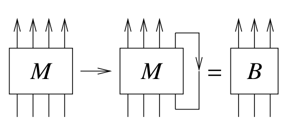


$a_{12}a_{13}a_{42}a_{12}$



$$\longrightarrow \quad \bar{y}_i = \bar{x}_i \qquad \bar{y}_j = (1 - T_i)\bar{x}_i + T_i\bar{x}_j$$

Linear Control Theory. If $\begin{pmatrix} y \\ y_n \end{pmatrix} = \begin{pmatrix} \Xi & \phi \\ \theta & \alpha \end{pmatrix}\begin{pmatrix} x \\ x_n \end{pmatrix}$, and we further impose $x_n = y_n$, then $y = Bx$ where $B = \Xi + \dfrac{\phi\theta}{1-\alpha}$. This fully explains the Gassner formulas and the GGA formula!



All that remains now is to replace TC by something more interesting: with $\epsilon^2 = 0$,

$$[a_{ij}, a_{ik}] = \epsilon(c_ja_{ik} - c_ka_{ij}).$$

Many further changes are also necessary, and the algebra is a lot more complicated and revolves around "quantization of Lie bialgebras" [EK, En]. But the spirit is right.

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Video and more at http://www.math.toronto.edu/~drorbn/Talks/NCSU-1604/