

The PBW Problem. In $\mathcal{U}(\mathfrak{g}^\epsilon)$, bring $Z = y^3 a^2 x^2 \cdot y^2 a^2 x$ to yax -order. In other words, find $g \in \mathbb{Z}[\epsilon, t, y, a, x]$ such that $Z = \mathcal{O}(f = y^3 y_2^2 a_1^2 a_2^2 x_1^2 x_2 : y_1 a_1 x_1 y_2 a_2 x_2) = \mathcal{O}(g : yax)$.

Solution, Part 1. In $\mathcal{U}(\mathfrak{g}^\epsilon)$ we have

$$X_{\tau_1, \eta_1, \alpha_1, \xi_1, \tau_2, \eta_2, \alpha_2, \xi_2} := e^{\tau_1 t} e^{\eta_1 y} e^{\alpha_1 a} e^{\xi_1 x} e^{\tau_2 t} e^{\eta_2 y} e^{\alpha_2 a} e^{\xi_2 x} \\ = e^{\tau t} e^{\eta y} e^{\alpha a} e^{\xi x} =: Y_{\tau, \eta, \alpha, \xi},$$

where τ, η, α, ξ are ugly functions of $\tau_1, \eta_1, \alpha_1, \xi_1$:

$$\tau = \tau_1 + \tau_2 - \frac{\log(1 - \epsilon \eta_2 \xi_1)}{\epsilon} = \tau_1 + \tau_2 + \eta_2 \xi_1 + \frac{\epsilon}{2} \eta_2^2 \xi_1^2 + \dots, \\ \eta = \eta_1 + \frac{e^{-\alpha_1} \eta_2}{(1 - \epsilon \eta_2 \xi_1)} = \eta_1 + e^{-\alpha_1} \eta_2 + \epsilon e^{-\alpha_1} \eta_2^2 \xi_1 + \dots, \\ \alpha = \alpha_1 + \alpha_2 + 2 \log(1 - \epsilon \eta_2 \xi_1) = \alpha_1 + \alpha_2 - 2 \epsilon \eta_2 \xi_1 + \dots, \\ \xi = \frac{e^{-\alpha_2} \xi_1}{(1 - \epsilon \eta_2 \xi_1)} + \xi_2 = e^{-\alpha_2} \xi_1 + \xi_2 + \epsilon e^{-\alpha_2} \eta_2 \xi_1^2 + \dots$$

Note 1. This defines a mapping $\Phi: \mathbb{R}_{\tau_1, \eta_1, \alpha_1, \xi_1, \tau_2, \eta_2, \alpha_2, \xi_2}^8 \rightarrow \mathbb{R}_{\tau, \eta, \alpha, \xi}^4$.

Proof. $\mathfrak{g}^\epsilon$ has a 2D representation ρ :

$$\rho t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \rho y = \begin{pmatrix} 0 & 0 \\ -\epsilon & 0 \end{pmatrix};$$

$$\rho a = \begin{pmatrix} (1 + 1/\epsilon) / 2 & 0 \\ 0 & -(1 - 1/\epsilon) / 2 \end{pmatrix}; \quad \rho x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};$$

$$\text{Simplify}[\{\rho a \cdot \rho x - \rho x \cdot \rho a = \rho x, \quad \rho a \cdot \rho y - \rho y \cdot \rho a = -\rho y, \\ \rho x \cdot \rho y - \rho y \cdot \rho x = \rho t - 2 \epsilon \rho a\}]$$

{True, True, True}

It is enough to verify the desired identity in ρ :

ME = MatrixExp;

Simplify[

$$\text{ME}[\tau_1 \rho t] \cdot \text{ME}[\eta_1 \rho y] \cdot \text{ME}[\alpha_1 \rho a] \cdot \text{ME}[\xi_1 \rho x] \cdot \text{ME}[\tau_2 \rho t] \cdot \\ \text{ME}[\eta_2 \rho y] \cdot \text{ME}[\alpha_2 \rho a] \cdot \text{ME}[\xi_2 \rho x] = \\ \text{ME}[\tau_0 \rho t] \cdot \text{ME}[\eta_0 \rho y] \cdot \text{ME}[\alpha_0 \rho a] \cdot \text{ME}[\xi_0 \rho x] / . \\ \left\{ \begin{aligned} \tau_0 &\rightarrow -\frac{\log[1 - \epsilon \eta_2 \xi_1]}{\epsilon} + \tau_1 + \tau_2, \quad \eta_0 \rightarrow \eta_1 + \frac{e^{-\alpha_1} \eta_2}{1 - \epsilon \eta_2 \xi_1}, \\ \alpha_0 &\rightarrow 2 \log[1 - \epsilon \eta_2 \xi_1] + \alpha_1 + \alpha_2, \quad \xi_0 \rightarrow \frac{e^{-\alpha_2} \xi_1}{1 - \epsilon \eta_2 \xi_1} + \xi_2 \end{aligned} \right\}$$

True

Solution, Part 2. But now, with $D_f = f(z \mapsto \partial_z) = \partial_{\eta_1}^3 \partial_{\alpha_1}^2 \partial_{\xi_1}^2 \partial_{\eta_2}^2 \partial_{\alpha_2}^2 \partial_{\xi_2}^2$,

$$Z = D_f X_{\tau_1, \eta_1, \alpha_1, \xi_1, \tau_2, \eta_2, \alpha_2, \xi_2} \Big|_{y=0} = D_f Y_{\tau, \eta, \alpha, \xi} \Big|_{y=0} \\ = \mathcal{O}\left(D_f e^{\tau t} e^{\eta y} e^{\alpha a} e^{\xi x} \Big|_{y=0} : yax\right) = \mathcal{O}(g : yax) :$$

$$\text{Expand}\left[\partial_{(\eta_1, 3)} \partial_{(\alpha_1, 2)} \partial_{(\xi_1, 2)} \partial_{(\eta_2, 2)} \partial_{(\alpha_2, 2)} \partial_{(\xi_2, 1)} \text{Exp}\left[\right. \right. \\ \left. \left. \left(-\frac{\log[1 - \epsilon \eta_2 \xi_1]}{\epsilon} + \tau_1 + \tau_2\right) t + \left(\eta_1 + \frac{e^{-\alpha_1} \eta_2}{1 - \epsilon \eta_2 \xi_1}\right) y + \right. \right. \\ \left. \left. (2 \log[1 - \epsilon \eta_2 \xi_1] + \alpha_1 + \alpha_2) a + \left(\frac{e^{-\alpha_2} \xi_1}{1 - \epsilon \eta_2 \xi_1} + \xi_2\right) x \right. \right. \\ \left. \left. \right] / . (\tau | \eta | \alpha | \xi)_{1|2} \rightarrow 0\right]$$

$$2 a^4 t^2 x y^3 + 4 t x^2 y^4 - 16 a t x^2 y^4 + 24 a^2 t x^2 y^4 - 16 a^3 t x^2 y^4 + \\ 4 a^4 t x^2 y^4 + 16 x^3 y^5 - 32 a x^3 y^5 + 24 a^2 x^3 y^5 - 8 a^3 x^3 y^5 + a^4 x^3 y^5 + \\ 2 a^4 t x y^3 - 8 a^5 t x y^3 + 8 x^2 y^4 - 40 a x^2 y^4 + 80 a^2 x^2 y^4 - \\ 80 a^3 x^2 y^4 + 40 a^4 x^2 y^4 - 8 a^5 x^2 y^4 - 4 a^5 x y^3 + 8 a^6 x y^3 + \epsilon^2$$

Note 2. Replacing $f \rightarrow D_f$ (and likewise $g \rightarrow D_g$), we find that $D_g = \Phi_* D_f$.

Note 3. The two great evils of mathematics are non-commutativity and non-linearity. We traded one for the other.

Note 4. We could have done similarly with $e^{\tau_1 t} e^{\eta_1 y} e^{\alpha_1 a} e^{\xi_1 x} = e^{\tau t + \eta y + \alpha a + \xi x}$, and with $S(e^{\tau_1 t} e^{\eta_1 y} e^{\alpha_1 a} e^{\xi_1 x})$, $\Delta(e^{\tau_1 t} e^{\eta_1 y} e^{\alpha_1 a} e^{\xi_1 x})$, $\prod_{i=1}^5 e^{\tau_i t} e^{\eta_i y} e^{\alpha_i a} e^{\xi_i x}$.

Fact. $R_{12} \rightarrow \exp(\partial_{\tau_1} \partial_{\alpha_2} + \partial_{y_1} \partial_{x_2})(1 + \sum_{d \geq 1} \epsilon^d p_d)$, where the p_d are computable polynomials of a-priori bounded degrees.

Moral. We need to understand the pushforwards via maps like Φ of (formally ∞ -order) “differential operators at 0”, that in themselves are perturbed Gaussians. This turns out to be the same problem as “0-dimensional QFT” (except no integration is ever needed), and if $\epsilon^{k+1} = 0$, it is explicitly soluble.

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dog·ma (dôg'mə, dôg'-)

The Free Dictionary, [oeß/TFD](#)

n. pl. dog·mas or dog·ma·ta (-mə-tə)

1. A doctrine or a corpus of doctrines relating to matters such as morality and faith, set forth in an authoritative manner by a religion.
2. A principle or statement of ideas, or a group of such principles or statements, especially when considered to be authoritative or accepted uncritically: “*Much education consists in the instilling of unfounded dogmas in place of a spirit of inquiry*” (Bertrand Russell).

diagram	n_k^+ Alexander’s ω^+ Today’s / Rozansky’s ρ_1^+	genus / ribbon unknotting number / amphicheiral	diagram	n_k^+ Alexander’s ω^+ Today’s / Rozansky’s ρ_1^+	genus / ribbon unknotting number / amphicheiral
	0_1^a 1 0	0 / ✓ 0 / ✓		3_1^a $t - 1$ t	1 / ✗ 1 / ✗
	4_1^a $3 - t$ 0	1 / ✗ 1 / ✓		5_1^a $t^2 - t + 1$ $2t^3 + 3t$	2 / ✗ 2 / ✗
	5_2^a $2t - 3$ $5t - 4$	1 / ✗ 1 / ✗		6_1^a $5 - 2t$ $t - 4$	1 / ✓ 1 / ✗