## Loading, initializing variables, setting default degree to 6.

(The *Mathematica* packages `FreeLie`` and `AwCalculus`` are at ωεβ/WKO4).

```
path = "C:/drorbn/AcademicPensieve/";
SetDirectory[path <> "2015-08/LesDiablerets-1508"];
Get[path <> "Projects/WKO4/FreeLie.m"];
Get[path <> "Projects/WKO4/AwCalculus.m"];
x = LW@"x"; y = LW@"y"; u = LW@"u";
$SeriesShowDegree = 6;
```

```
FreeLie` implements / extends
 {*, +, **, $SeriesShowDegree, ⟨⟩, ∫, ≡, ad, Ad, adSeries, AllCyclicWords,
  AllLyndonWords, AllWords, Arbitrator, ASeries, AW, b, BCH, BooleanSequence,
  BracketForm, BS, CC, Crop, cw, CW, CWS, CWSeries, D, Deg, DegreeScale,
  DerivationSeries, div, DK, DKS, DKSeries, EulerE, Exp, Inverse, j, J, JA,
  LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization, Morphism,
  New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve, Support, t,
  tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ι, Λ, σ, ℏ, ⌒, ⌢}.

FreeLie` is in the public domain. Dror Bar-Natan is committed to
  support it within reason until July 15, 2022. This is version 150814.

AwCalculus` implements / extends
 {*, **, ≡, dA, dc, deg, dm, dS, dΔ, dη, dσ, El, Es, hA, hm, hS, hΔ, hη,
  hσ, RandomElSeries, RandomEsSeries, tA, tha, tm, tS, tΔ, tη, tσ, Γ, Λ}.

AwCalculus` is in the public domain. Dror Bar-Natan is committed to
  support it within reason until July 15, 2022. This is version 150814.
```

```
BCH[x, y] (* Can raise degree to 22 *)
```

$$\mathrm{LS}\Big[\overline{x}+\overline{y},\ \tfrac{\overline{xy}}{2},\ \tfrac{1}{12}\overline{x\overline{xy}}+\tfrac{1}{12}\overline{\overline{xy}y},\ \tfrac{1}{24}\overline{x\overline{\overline{xy}y}},\ -\tfrac{1}{720}\overline{x\overline{x\overline{x\overline{xy}}}}+\tfrac{1}{180}\overline{x\overline{x\overline{\overline{xy}y}}}+\tfrac{1}{180}\overline{x\overline{\overline{\overline{xy}y}y}}+\tfrac{1}{120}\overline{\overline{xy}\overline{\overline{xy}y}}+\tfrac{1}{360}\overline{\overline{x\overline{xy}}\overline{xy}}-\tfrac{1}{720}\overline{\overline{\overline{\overline{xy}y}y}y},\ -\tfrac{\overline{xxx\overline{xy}}}{1440}+\tfrac{1}{360}\overline{xx\overline{\overline{xy}y}y}+\tfrac{1}{240}\overline{x\overline{xy}\overline{\overline{xy}y}}+\tfrac{1}{720}\overline{x\overline{x\overline{xy}}\overline{xy}}-\tfrac{\overline{x\overline{xy}yyy}}{1440},\ \ldots\Big]$$

## KV Direct.

```
{F = LS[{x, y}, Fs], G = LS[{x, y}, Gs]}; Fs["y"] = 1/2;
SeriesSolve[{F, G},
  ℏ^-1 (LS[x + y] - BCH[y, x] ≡ F - G - Ad[-x][F] + Ad[y][G]) ⋀
   div_x[F] + div_y[G] ≡
    1/2 tr_u[adSeries[ad/(e^ad - 1), x][u] + adSeries[ad/(e^ad - 1), y][u] -
      adSeries[ad/(e^ad - 1), BCH[x, y]][u]]];
{F, G} (* Can raise degree to 13 *)
```

$$\Big\{\mathrm{LS}\Big[\tfrac{\overline{y}}{2},\ \tfrac{\overline{xy}}{6},\ \tfrac{1}{24}\overline{\overline{xy}y},\ -\tfrac{1}{180}\overline{x\overline{x\overline{xy}}}+\tfrac{1}{80}\overline{x\overline{\overline{xy}y}}+\tfrac{1}{360}\overline{\overline{\overline{xy}y}y},\ -\tfrac{1}{720}\overline{xx\overline{\overline{xy}y}}+\tfrac{1}{240}\overline{x\overline{\overline{xy}y}y}+\tfrac{1}{240}\overline{\overline{xy}\overline{\overline{xy}y}}+\tfrac{1}{720}\overline{\overline{x\overline{xy}}\overline{xy}}-\tfrac{\overline{\overline{xy}yyy}}{1440},\ \tfrac{\overline{xxx\overline{xy}}}{5040}-\tfrac{\overline{xxx\overline{xy}}}{1344}+\tfrac{13\,\overline{xx\overline{xy}yy}}{15120}+\tfrac{1}{840}\overline{x\overline{xy}\overline{xy}y}+\tfrac{\overline{x\overline{xy}\overline{xy}}}{3360}+\tfrac{\overline{x\overline{xy}yy}}{6720}+\tfrac{\overline{\overline{xy}\overline{xy}yy}}{1260}+\tfrac{\overline{x\overline{xy}\overline{xy}}}{1680}-\tfrac{\overline{\overline{xy}yyyy}}{10080},\ \ldots\Big],$$
$$\mathrm{LS}\Big[0,\ \tfrac{\overline{xy}}{12},\ \tfrac{1}{24}\overline{\overline{xy}y},\ -\tfrac{1}{360}\overline{x\overline{x\overline{xy}}}+\tfrac{1}{120}\overline{x\overline{\overline{xy}y}}+\tfrac{1}{180}\overline{\overline{\overline{xy}y}y},\ -\tfrac{1}{720}\overline{xx\overline{\overline{xy}y}}+\tfrac{1}{240}\overline{x\overline{\overline{xy}y}y}+\tfrac{1}{240}\overline{\overline{xy}\overline{\overline{xy}y}}+\tfrac{1}{720}\overline{\overline{x\overline{xy}}\overline{xy}}-\tfrac{\overline{\overline{xy}yyy}}{1440},\ \tfrac{\overline{xxx\overline{xy}}}{10080}-\tfrac{\overline{xxx\overline{xy}}}{2016}+\tfrac{\overline{xx\overline{xy}y}}{1890}+\tfrac{\overline{x\overline{xy}\overline{xy}}}{1120}+\tfrac{\overline{x\overline{x\overline{xy}}\overline{xy}}}{5040}+\tfrac{\overline{x\overline{xy}yy}}{2520}+\tfrac{1}{840}\overline{\overline{xy}\overline{\overline{xy}y}y}+\tfrac{\overline{x\overline{xy}\overline{xy}}}{1260}-\tfrac{\overline{\overline{xy}yyy}}{5040},\ \ldots\Big]\Big\}$$

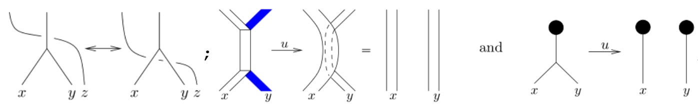
## Meaningless calculations.

```
{b[F, G], tr_x[F]}
```

$$\Big\{\mathrm{LS}\Big[0,\ 0,\ -\tfrac{1}{24}\overline{\overline{xy}y},\ -\tfrac{1}{48}\overline{\overline{\overline{xy}y}y},\ \tfrac{1}{720}\overline{xx\overline{xy}}-\tfrac{1}{240}\overline{x\overline{\overline{xy}y}y}-\tfrac{\overline{xy\overline{xy}y}}{1440}-\tfrac{1}{720}\overline{\overline{x\overline{xy}}\overline{xy}}-\tfrac{1}{360}\overline{\overline{\overline{xy}y}yy},\ \tfrac{\overline{xx\overline{xy}yy}}{1440}-\tfrac{1}{480}\overline{x\overline{\overline{xy}y}yy}-\tfrac{1}{288}\overline{\overline{xy}\overline{\overline{xy}y}y}-\tfrac{7\,\overline{x\overline{xy}\overline{xy}}}{2880}+\tfrac{\overline{\overline{xy}yyy}}{2880},\ \ldots\Big],$$
$$\mathrm{CWS}\Big[-\tfrac{\overline{y}}{6},\ \tfrac{\overline{yy}}{24},\ \tfrac{\overline{xxy}}{180}+\tfrac{\overline{xyy}}{80}-\tfrac{\overline{yyy}}{360},\ -\tfrac{\overline{xxyy}}{180}+\tfrac{\overline{xyxy}}{240}-\tfrac{\overline{xyyy}}{240}-\tfrac{\overline{yyyy}}{1440},\ -\tfrac{\overline{xxxxy}}{5040}+\tfrac{\overline{xxxyy}}{6720}-\tfrac{\overline{xxyxy}}{1120}+\tfrac{2\,\overline{xxyyy}}{945}-\tfrac{\overline{xyxyy}}{336}+\tfrac{\overline{xyyyy}}{6720}+\tfrac{\overline{yyyyy}}{10080},\ \tfrac{\overline{xxxxyy}}{3360}-\tfrac{\overline{xxxyxy}}{1344}-\tfrac{\overline{xxxyyy}}{2240}+\tfrac{\overline{xxyxxy}}{2016}+\tfrac{13\,\overline{xxyxyy}}{10080}+\tfrac{\overline{xxyyxy}}{1680}-\tfrac{\overline{xxyyyy}}{3780}-\tfrac{\overline{xyxyxy}}{840}+\tfrac{\overline{xyxyyy}}{5040}+\tfrac{\overline{xyyxyy}}{2240}+\tfrac{\overline{xyyyyy}}{6720}+\tfrac{\overline{yyyyyy}}{60480},\ \ldots\Big]\Big\}$$

(Also implemented: $\partial_\lambda$ and derivations in general, `tb`, $e^{\partial_\lambda}$ and morphisms in general, `div`, `j`, Drinfel'd-Kohno, etc.)

## The [BND] "vertex" equations.




```
α = LS[{x, y}, αs]; β = LS[{x, y}, βs];
γ = CWS[{x, y}, γs];
V = Es[⟨x → α, y → β⟩, γ];
κ = CWS[{x}, κs]; Cap = Es[⟨x → LS[0]⟩, κ];
Rs[a_, b_] := Es[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
R4Eqn = V ** (Rs[x, z] // dΔ[x, x, y]) ≡ Rs[y, z] ** Rs[x, z] ** V;
UnitarityEqn =
  (V ** (V // dA) ≡ Es[⟨x → LS[0], y → LS[0]⟩, CWS[0]]);
CapEqn = ((V ** (Cap // dΔ[x, x, y]) // dc[x] // dc[y]) ≡
    (Cap (Cap // dσ[x, y]) // dc[x] // dc[y]));
βs["x"] = 1/2; βs["y"] = 0;
SeriesSolve[{α, β, γ, κ},
  (ℏ^-1 R4Eqn) ⋀ UnitarityEqn ⋀ CapEqn];
{V, κ}
```

SeriesSolve::ArbitrarilySetting : In degree 1 arbitrarily setting {κs[x] → 0}.
SeriesSolve::ArbitrarilySetting : In degree 3 arbitrarily setting {αs[x, y, y] → 0}.
SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {αs[x, x, x, y, y] → 0}.
General::stop : Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. ≫

$$\Big\{\mathrm{Es}\Big[\Big\langle \overline{x}\to\mathrm{LS}\Big[0,\ -\tfrac{\overline{xy}}{24},\ 0,\ \tfrac{7\,\overline{x\overline{x\overline{xy}}}}{5760}-\tfrac{7\,\overline{x\overline{\overline{xy}y}}}{5760}+\tfrac{\overline{\overline{\overline{xy}y}y}}{1440},\ 0,\ -\tfrac{31\,\overline{xxx\overline{xy}}}{967680}+\tfrac{31\,\overline{xxx\overline{xy}}}{483840}-\tfrac{83\,\overline{xx\overline{xy}y}}{967680}-\tfrac{31\,\overline{x\overline{xy}\overline{xy}}}{725760}-\tfrac{31\,\overline{x\overline{xy}\overline{xy}}}{645120}+\tfrac{13\,\overline{x\overline{xy}yy}}{241920}+\tfrac{101\,\overline{\overline{xy}\overline{xy}y}}{1451520}+\tfrac{527\,\overline{x\overline{xy}\overline{xy}}}{5806080}-\tfrac{\overline{\overline{xy}yyy}}{60480},\ \ldots\Big],$$
$$\overline{y}\to\mathrm{LS}\Big[\tfrac{\overline{x}}{2},\ -\tfrac{\overline{xy}}{12},\ 0,\ \tfrac{\overline{x\overline{x\overline{xy}}}}{5760}-\tfrac{1}{720}\overline{x\overline{\overline{xy}y}}+\tfrac{1}{720}\overline{\overline{\overline{xy}y}y},\ -\tfrac{\overline{xx\overline{xy}}}{7680}+\tfrac{\overline{xx\overline{xy}}}{3840}-\tfrac{\overline{\overline{xy}\overline{xy}}}{6912},\ -\tfrac{\overline{xxx\overline{xy}}}{645120}+\tfrac{23\,\overline{xxx\overline{xy}}}{483840}-\tfrac{13\,\overline{xx\overline{xy}y}}{161280}-\tfrac{\overline{x\overline{xy}\overline{xy}}}{22680}-\tfrac{41\,\overline{x\overline{xy}\overline{xy}}}{580608}+\tfrac{\overline{x\overline{xy}yy}}{15120}+\tfrac{\overline{\overline{xy}\overline{xy}y}}{12096}+\tfrac{71\,\overline{x\overline{xy}\overline{xy}}}{483840}-\tfrac{\overline{\overline{xy}yyy}}{30240},\ \ldots\Big]\Big\rangle,$$
$$\mathrm{CWS}\Big[0,\ -\tfrac{\overline{xy}}{48},\ 0,\ \tfrac{\overline{xxxy}}{2880}+\tfrac{\overline{xxyy}}{2880}+\tfrac{\overline{xyxy}}{5760}+\tfrac{\overline{xyyy}}{2880},\ 0,\ -\tfrac{\overline{xxxxxy}}{120960}-\tfrac{\overline{xxxxyy}}{120960}-\tfrac{\overline{xxxyxy}}{120960}-\tfrac{\overline{xxxyyy}}{120960}-\tfrac{\overline{xxyxxy}}{241920}-\tfrac{\overline{xxyxyy}}{120960}-\tfrac{\overline{xxyyxy}}{120960}-\tfrac{\overline{xxyyyy}}{120960}-\tfrac{\overline{xyxyxy}}{362880}-\tfrac{\overline{xyxyyy}}{120960}-\tfrac{\overline{xyyxyy}}{241920}-\tfrac{\overline{xyyyyy}}{120960},\ \ldots\Big]\Big],$$
$$\mathrm{CWS}\Big[0,\ -\tfrac{\overline{xx}}{96},\ 0,\ \tfrac{\overline{xxxx}}{11520},\ 0,\ -\tfrac{\overline{xxxxxx}}{725760},\ \ldots\Big]\Big\}$$

Video and more at http://www.math.toronto.edu/~drorbn/Talks/LesDiablerets-1508/